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The Dynamical Yang-Baxter Equation, Representation Theory, and Quantum Integrable Systems

Pavol Etingof and Frédéric Latour

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The Dynamical Yang-Baxter Equation, Representation Theory, and Quantum Integrable Systems

Pavel Etingof and Frédéric Latour

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PREFACE

This book is based on the second author's notes of the first author's graduate course given at MIT in the Fall of 2001. It gives an introduction to the theory of the dynamical Yang–Baxter equation and its applications.

The dynamical Yang–Baxter equations (both classical and quantum) appeared first in physical literature (Gervais and Neveu (1984), Faddeev (1990), Balog *et al.* (1990)). They are meaningful generalizations of the usual Yang–Baxter equations, in which the unknown is not a matrix, but rather a matrix-valued function of a "dynamical" parameter λ taking values in some abelian group. Unlike the usual Yang–Baxter equations (which are algebraic), the classical dynamical Yang–Baxter equation is a differential equation, and the quantum dynamical Yang–Baxter equation is a difference equation. This makes them richer and more interesting than the usual Yang–Baxter equations.

An intensive mathematical study of the dynamical Yang–Baxter equations began with the innovative work of Felder (1994), where it was explained how these equations and their solutions (dynamical R-matrices) naturally arise in conformal field theory and statistical mechanics, and also how to attach to any solution of the quantum dynamical Yang–Baxter equation a certain quantum group (together with the corresponding tensor category of representations). This gave a beginning to a vast and vibrant new field. The papers that followed the work (Felder (1994)) unearthed the geometric and representation theoretic meaning of the dynamical Yang–Baxter equations, as well as their numerous connections with other mathematical subjects (integrable systems, special functions). A review of some of these works can be found in Etingof and Schiffmann (2001*a*), Etingof (2002).

The goal of this book is to give an introduction to just one exciting part of the theory of the dynamical Yang–Baxter equations, namely the connections of the quantum dynamical Yang–Baxter equation with representation theory of semisimple Lie algebras and quantum groups, and with integrable systems of Macdonald–Ruijsenaars type. Thus the book does not attempt to review the whole theory. We hope that this is, to some extent, expiated by the fact that it starts from scratch and contains many detailed proofs and explicit calculations. This should make the book accessible to beginners, who are familiar with the basics of representation theory of semisimple Lie algebras.

The composition of the book is as follows.

Chapter 1 gives an introduction to the subject, highlighting the significance of the dynamical Yang–Baxter equation for representation theory and mathematical physics. Here we briefly review the theory of the dynamical Yang–Baxter equation, to be presented in the subsequent chapters, and explain the connections of the dynamical Yang–Baxter equation with other objects in the theory of representations and quantum integrable systems. In this chapter we take the opportunity to briefly discuss the topics which could not be included in the book, such as dynamical quantum groups and their representations.

In Chapter 2, we review, without proofs, the background material about semisimple Lie algebras.

In Chapter 3, we introduce the main characters: intertwiners, fusion operators, and exchange operators. Further, we establish the main properties of these objects: the dynamical twist equation for the fusion operator and the quantum dynamical Yang–Baxter equation for the exchange operator. In this way, we show that the quantum dynamical Yang–Baxter equation arises naturally in representation theory of semisimple Lie algebras. Finally, we show that the fusion operator satisfies the Arnaudon–Buffenoir–Ragoucy–Roche equation, and using this equation compute this operator in the $\mathfrak{sl}_2$-case.

In Chapter 4, we review the theory of quantum groups $\mathfrak{U}_\mathfrak{q}(\mathfrak{g})$. At the end of the Chapter, motivated by the theory of quantum groups, we discuss the classical dynamical Yang–Baxter equation, which is the classical limit of the quantum dynamical Yang–Baxter equation.

In Chapter 5, we generalize the constructions of Chapter 2 to the case of quantum groups.

In Chapter 6, we discuss classical and quantum integrable systems, and the transfer matrix construction, which allows one to attach a quantum integrable system to an R-matrix. Then we generalize the transfer matrix construction to dynamical R-matrices. When applied to the exchange matrix, this construction yields an integrable system which is called the Macdonald–Ruijsenaars system.

In Chapter 7, we introduce traces of intertwining operators, and show that they are eigenfunctions of the (modified) Macdonald–Ruijsenaars system. We also show that they are symmetric with respect to the two weights on which they depend (the bispectrality property). We also compute the traces in the $\mathfrak{sl}_2$ case.

In Chapter 8, we consider the special case where eigenfunctions of the Macdonald–Ruijsenaars system specialize to Macdonald functions and polynomials, and rederive (using the theory of dynamical R-matrices) the results of Etingof–Kirillov on the representation theoretic interpretation of Macdonald polynomials of type A.

Finally, in Chapter 9 we give an introduction to the theory of dynamical Weyl group, developed by Tarasov, Varchenko, and the first author.

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INTRODUCTION

1.1 The quantum dynamical Yang–Baxter equation

1.1.1 The equation

The quantum dynamical Yang–Baxter equation (QDYBE) was first considered by physicists Gervais and Neveu, and later studied systematically by Felder. It is an equation with respect to a (meromorphic) function $R : \mathfrak{h}^* \to \mathsf{End}_{\mathfrak{h}}(V \otimes V)$, where $\mathfrak{h}$ is a commutative finite dimensional Lie algebra over $\mathbf{C}$, and V is a semisimple $\mathfrak{h}$-module. It reads

$$R^{12}(\lambda - h^3)R^{13}(\lambda)R^{23}(\lambda - h^1) = R^{23}(\lambda)R^{13}(\lambda - h^2)R^{12}(\lambda)$$

on $V \otimes V \otimes V$. Here h^i is the *dynamical notation*, to be extensively used below: for instance, $R^{12}(\lambda - h^3)$ is defined by the formula $R^{12}(\lambda - h^3)(v_1 \otimes v_2 \otimes v_3) \stackrel{\text{def}}{=} \big(R^{12}(\lambda - \mu)(v_1 \otimes v_2)\big) \otimes v_3$ if v_3 is of weight μ under $\mathfrak{h}$.

Invertible solutions of QDYBE are called quantum dynamical R-matrices. If $\mathfrak{h} = 0$, QDYBE turns into the usual quantum Yang–Baxter equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$

1.1.2 Examples of solutions of QDYBE

We will soon explain the origin and meaning of QDYBE. Now let us consider some examples of its solutions. Let V be the vector representation of $\mathfrak{sl}_n$, and $\mathfrak{h}$ the Lie algebra of traceless diagonal matrices. In this case $\lambda \in \mathfrak{h}^*$ can be written as $\lambda = (\lambda_1, ..., \lambda_n)$, where $\lambda_i \in \mathbf{C}$. Let v_a, $a = 1, ..., n$ be the standard basis of V. Let E_{ab} be the matrix units given by $E_{ab}v_c = \delta_{bc}v_a$.

We will now give a few examples of quantum dynamical R-matrices. The general form of the R-matrices will be

$$R = \sum_a E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab} E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab} E_{ab} \otimes E_{ba}, \qquad (1.1)$$

where α_{ab}, β_{ab} are functions which will be given explicitly in each example.

Example 1.1 The basic rational solution. Let

$$\beta_{ab} = 1/(\lambda_b - \lambda_a), \alpha_{ab} = 1 + \beta_{ab}.$$

Then $R(\lambda)$ is a dynamical R-matrix.

Example 1.2 The basic trigonometric solution. Let

$$\beta_{ab} = (\mathsf{q} - 1)/(\mathsf{q}^{\lambda_b - \lambda_a} - 1), \alpha_{ab} = \mathsf{q} + \beta_{ab}.$$

Then $R(\lambda)$ is a dynamical R-matrix.

Remark 1.3 In Examples 1.1 and 1.2, the dynamical R-matrix satisfies the *Hecke condition* $(PR - 1)(PR + \mathsf{q}) = 0$, with $\mathsf{q} = 1$ in Example 1.1 (where P is the permutation on $V \otimes V$).

Remark 1.4 The basic trigonometric solution degenerates into the basic rational solution as $\mathsf{q} \to 1$.

1.1.3 The QDYBE with spectral parameter

We note that the QDYBE has an important generalization, which is the *QDYBE with spectral parameter*. It is an equation with respect to a meromorphic function $R : \mathbf{C} \times \mathfrak{h}^* \to \mathsf{End}_{\mathfrak{h}}(V \otimes V)$ which reads

$$\begin{aligned} &R^{12}(u_{12}, \lambda - h^3)R^{13}(u_{13}, \lambda)R^{23}(u_{23}, \lambda - h^1) \\ &\quad = R^{23}(u_{23}, \lambda)R^{13}(u_{13}, \lambda - h^2)R^{12}(u_{12}, \lambda), \quad \text{where } u_{ij} = u_i - u_j. \end{aligned}$$

This equation (as well as its basic solution, the so-called elliptic solution (Felder (1994))) is even more useful than the QDYBE itself. However, to keep this book short, we are forced to skip this topic; the inquisitive reader is referred to Felder (1994), Etingof (2002) and references therein for more details.

1.1.4 The tensor category of representations associated to a quantum dynamical R-matrix

Let R be a quantum dynamical R-matrix with spectral parameter. According to Felder (1994), a representation of R is a semisimple $\mathfrak{h}$-module W and an invertible meromorphic function $L = L_W : \mathfrak{h}^* \to \mathsf{End}_{\mathfrak{h}}(V \otimes W)$, such that

$$R^{12}(\lambda - h^3)L^{13}(\lambda)L^{23}(\lambda - h^1) = L^{23}(\lambda)L^{13}(\lambda - h^2)R^{12}(\lambda). \tag{1.2}$$

For example: $(\mathbf{C}, 1)$ (trivial representation) and (V, R) (vector representation).

A morphism $f : (W, L_W) \to (W', L_{W'})$ is a meromorphic function $f : \mathfrak{h}^* \to \mathsf{End}_{\mathfrak{h}}(W)$ such that

$$(1 \otimes f(\lambda))L_W(\lambda) = L_{W'}(\lambda)(1 \otimes f(\lambda - h^1)).$$

With this definition, representations form an (additive) category $\mathsf{Rep}(R)$. Moreover, it is a tensor category (Felder (1994)): given (W, L_W) and (U, L_U), one can form the tensor product representation $(W \otimes U, L_{W \otimes U})$, where

$$L_{W \otimes U}(u, \lambda) = L_W^{12}(\lambda - h^3)L_U^{13}(\lambda);$$

tensor product of morphisms is defined by $(f \otimes g)(\lambda) = f(\lambda - h^2) \otimes g(\lambda)$.

1.1.5 Gauge transformations and classification

There exists a group of rather trivial transformations acting on quantum dynamical R-matrices. They are called *gauge transformations.* If $\mathfrak{h}$ and V are as in the previous section, then gauge transformations are:

1. Twist by a closed multiplicative 2-form ϕ: $\alpha_{ab} \mapsto \alpha_{ab}\phi_{ab}$, where $\phi_{ab} = \phi_{ba}^{-1}$, and
$$\phi_{ab}(\lambda)\phi_{bc}(\lambda)\phi_{ca}(\lambda) = \phi_{ab}(\lambda - \omega_c)\phi_{bc}(\lambda - \omega_a)\phi_{ca}(\lambda - \omega_b)$$
($\omega_i = \text{weight}(v_i)$);
2. Permutation of indices $a = 1, ..., n$; $\lambda \mapsto \lambda - \nu$.

Theorem 1.5 (Etingof and Varchenko (1998*b*)) *Any quantum dynamical R-matrix for* $\mathfrak{h}, V$ *satisfying the Hecke condition with* $\mathsf{q} = 1$ *(respectively,* $\mathsf{q} \neq 1$*) is a gauge transformation of the basic rational (respectively, trigonometric) solution, or a limit of such R-matrices.*

Remark 1.6 Gauge transformations 2 do not affect the representation category of the R-matrix. Gauge transformation 1 does not affect the category if the closed form ϕ is exact: $\phi = d\xi$, i.e.,

$$\phi_{ab}(\lambda) = \xi_a(\lambda)\xi_b(\lambda - \omega_a)\xi_a(\lambda - \omega_b)^{-1}\xi_b(\lambda)^{-1},$$

where $\xi_a(\lambda)$ is a collection of meromorphic functions.

1.1.6 Dynamical quantum groups

Equation (1.2) may be regarded as a set of defining relations for an associative algebra A_R (see Etingof and Varchenko (1998*b*) for precise definitions). This algebra is a dynamical analog of the quantum group attached to an R-matrix defined in Feddeev *et al.* (1988), and representations of R are an appropriate class of representations of this algebra. The algebra A_R is called the dynamical quantum group attached to R.

To keep this book within bounds, we will not discuss A_R in detail. However, let us mention (Etingof and Varchenko (1998*b*)) that A_R is a bialgebroid with base $\mathfrak{h}^*$. This corresponds to the fact that the category $\mathsf{Rep}(R)$ is a tensor category. Moreover, if R satisfies an additional rigidity assumption (valid for example for the basic rational and trigonometric solutions) then the category $\mathsf{Rep}(R)$ has duality, and A_R is a Hopf algebroid, or a quantum groupoid (i.e. it has an antipode).

Remark 1.7 For a general theory of bialgebroids and Hopf algebroids the reader is referred to Lu (1996). However, let us mention that bialgebroids with base X correspond to pairs (tensor category, tensor functor to $O(X)$-bimodules), similarly to how bialgebras correspond to pairs (tensor category, tensor functor to vector spaces) (i.e. via Tannakian formalism).

1.1.7 The classical dynamical Yang–Baxter equation

In the theory of quantum groups, an important role is played by the following fact: if $R = 1 - \hbar r + O(\hbar^2)$ is a solution of QYBE, then r satisfies the classical Yang–Baxter equation (CYBE),

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

The element r is called the classical limit of R, while R is called a quantization of r.

The dynamical analog of this is the following. First of all define the *QDYBE with step* $\gamma \in \mathbf{C}^*$, which differs from the usual QDYBE by the replacement $h^i \to \gamma h^i$. Clearly, $R(\lambda)$ satisfies QDYBE if and only if $R(\lambda/\gamma)$ satisfies QDYBE with step γ.

Now let $R(\lambda, \hbar)$ be a family of solutions of QDYBE with step $\hbar$ given by a series $1 - \hbar r(\lambda) + O(\hbar^2)$. Then it is easy to show that $r(\lambda)$ satisfies the following differential equation, called the *classical dynamical Yang–Baxter equation* (CDYBE) (Balog *et al.* (1990), Felder (1994)):

$$\sum_i \left(\mathsf{x}_i^{(1)} \frac{\partial r^{23}}{\partial \mathsf{x}_i} - \mathsf{x}_i^{(2)} \frac{\partial r^{13}}{\partial \mathsf{x}_i} + \mathsf{x}_i^{(3)} \frac{\partial r^{12}}{\partial \mathsf{x}_i} \right) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \tag{1.3}$$

where x_i is a basis of $\mathfrak{h}$. The function $r(\lambda)$ is called the classical limit of $R(\lambda, \hbar)$, and $R(\lambda, \hbar)$ is called a quantization of $r(\lambda)$.

Define a classical dynamical r-matrix to be a meromorphic function $r : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ satisfying the CDYBE.

Conjecture 1.8 *Any classical dynamical r-matrix can be quantized.*

This conjecture is proved in Etingof and Kazhdan (1996) for the non-dynamical case, and in Xu (2002) for the dynamical case for skew-symmetric solutions ($r^{21} = -r$) satisfying additional technical assumptions. However, the most interesting non-skew-symmetric case is still open.

Remark 1.9 Similarly to CYBE, CDYBE makes sense for functions with values in $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra containing $\mathfrak{h}$.

Remark 1.10 The classical limit of the notion of a quantum groupoid is the notion of a Poisson groupoid, due to Weinstein. By definition, a Poisson groupoid is a groupoid G which is also a Poisson manifold, such that the graph of the multiplication is coisotropic in $G \times G \times \bar{G}$ (where $\bar{G}$ is G with reversed sign of Poisson bracket). Such a groupoid can be attached (Etingof and Varchenko (1998*a*)) to a classical dynamical r-matrix $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$, such that $r^{21} + r$ is constant and invariant (i.e. r is a "dynamical quasitriangular structure" on $\mathfrak{g}$). This is the classical limit of the assignment of a quantum groupoid to a quantum dynamical R-matrix (Etingof and Varchenko (1998*b*)).

1.1.8 Examples of solutions of CDYBE

We will now give examples of solutions of CDYBE in the case when $\mathfrak{g}$ is a finite dimensional simple Lie algebra, and $\mathfrak{h}$ is its Cartan subalgebra. We fix an invariant inner product on $\mathfrak{g}$. It is restricted to a nondegenerate inner product on $\mathfrak{h}$. Using this inner product, we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ ($\lambda \in \mathfrak{h}^* \mapsto \bar{\lambda} \in \mathfrak{h}$), which yields an inner product on $\mathfrak{h}^*$. The normalization of the inner product is chosen so that short roots have squared length 2. Let x_i be an orthonormal basis of $\mathfrak{h}$, and let e_α, $\mathsf{e}_{-\alpha}$ denote positive (respectively, negative) root elements of $\mathfrak{g}$, such that $\langle \mathsf{e}_\alpha, \mathsf{e}_{-\alpha} \rangle = 1$.

Here are some examples of solutions of CDYBE (see section 5.4 for a more complete discussion).

Example 1.11 The basic rational solution is

$$r(\lambda) = \sum_{\alpha>0} \frac{\mathsf{e}_\alpha \wedge \mathsf{e}_{-\alpha}}{\langle \lambda, \alpha \rangle}.$$

Example 1.12 The basic trigonometric solution is

$$r(\lambda) = \frac{\Omega}{2} + \sum_{\alpha>0} \frac{1}{2}\, \mathsf{e}_\alpha \wedge \mathsf{e}_{-\alpha} \,\mathsf{cotanh}\left(\frac{\langle \lambda, \alpha \rangle}{2}\right),$$

where $\Omega \in S^2\mathfrak{g}$ is the inverse element to the inner product on $\mathfrak{g}$.

Remark 1.13 One says that a classical dynamical r-matrix r has coupling constant ε if $r + r^{21} = \varepsilon\Omega$ (this is an classical analog of the Hecke condition in the quantum case). With these definitions, the basic rational solution has coupling constant 0, while the trigonometric solution has coupling constant 1.

Remark 1.14 The classical limit of the basic rational and trigonometric solution of QDYBE (modified by $\lambda \mapsto \lambda/\hbar$) is the basic rational, respectively trigonometric, solution of CDYBE for $\mathfrak{g} = \mathfrak{sl}_n$ (in the trigonometric case we should set $\mathsf{q} = e^{-\hbar/2}$).

Remark 1.15 These examples make sense for any reductive Lie algebra $\mathfrak{g}$.

1.1.9 Gauge transformations and classification of solutions for CDYBE

It is clear from the above that it is interesting to classify solutions of CDYBE. As in the quantum case, it should be done up to gauge transformations. These transformations are classical analogs of the gauge transformations in the quantum case. They are the following:

1. $r \mapsto r + \omega$, where $\omega = \sum_{i,j} C_{ij}(\lambda)\, \mathsf{x}_i \wedge \mathsf{x}_j$ is a meromorphic closed differential 2-form on $\mathfrak{h}^*$.
2. $r(\lambda) \mapsto ar(a\lambda - \nu)$; Weyl group action.

Theorem 1.16 (Etingof and Varchenko (1998*a*))

1. *Any classical dynamical r-matrix with zero coupling constant is a gauge transformation of the basic rational solution for a reductive subalgebra* $\mathfrak{l}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{h}$*, or its limiting case.*
2. *Any classical dynamical r-matrix with nonzero coupling constant is a gauge transformation of the basic trigonometric solution for* $\mathfrak{g}$*, or its limiting case.*

Remark 1.17 One may also classify dynamical r-matrices with nonzero coupling constant defined on $\mathfrak{l}^*$ for a Lie subalgebra $\mathfrak{l} \subset \mathfrak{h}$, on which the inner product is nondegenerate (Schiffmann (1998)). Up to gauge transformations they are classified by generalized Belavin–Drinfeld triples, i.e.triples (Γ_1, Γ_2, T), where Γ_i are subdiagrams of the Dynkin diagram Γ of $\mathfrak{g}$, and $T : \Gamma_1 \to \Gamma_2$ is a bijection preserving the inner product of simple roots (so this classification is a dynamical analog of the Belavin–Drinfeld classification of r-matrices on simple Lie algebras, and the classification of Etingof and Varchenko (1998*a*) is the special case $\Gamma_1 = \Gamma_2 = \Gamma$, $T = \mathsf{Id}$). Explicit quantization of the dynamical r-matrices from Schiffmann (1998) is given in Etingof *et al.* (2000).

1.2 The fusion and exchange construction

It is striking that unlike QYBE, interesting solutions of QDYBE may be obtained already from classical representation theory of Lie algebras. This can be done through the fusion and exchange construction (see Faddeev (1990), Etingof and Varchenko (1999)).

1.2.1 Intertwining operators

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbf{C}$, with polar decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. For any $\mathfrak{g}$-module V, we write $V[\nu]$ for the weight subspace of V of weight $\nu \in \mathfrak{h}^*$. Let M_λ denote the Verma module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$, x_λ being its highest weight vector, and x_λ^* the lowest weight vector of the dual module. Let V be a finite-dimensional representation of $\mathfrak{g}$. Consider an intertwining operator $\Phi : M_\lambda \to M_\mu \otimes V$. The vector $x_\mu^*(\Phi x_\lambda) \in V[\lambda - \mu]$ is called the expectation value of Φ, and denoted by $\langle \Phi \rangle$.

Lemma 1.18 *If* M_μ *is irreducible (i.e. for generic* μ*), the map* $\Phi \to \langle \Phi \rangle$ *is an isomorphism* $\mathsf{Hom}_{\mathfrak{g}}(M_{\mu+\nu}, M_\mu \otimes V) \to V[\nu]$.

Lemma 1.18 allows one to define for any $v \in V[\nu]$ (and generic λ) the intertwining operator $\Phi_\lambda^v : M_\lambda \to M_{\lambda-\nu} \otimes V$, such that $\langle \Phi_\lambda^v \rangle = v$.

1.2.2 The fusion and exchange operators

Now let V, W be finite-dimensional $\mathfrak{g}$-modules, and $v \in V, w \in W$ homogeneous vectors, of weights $\mathsf{wt}\, v, \mathsf{wt}\, w$. Consider the composition of two intertwining operators

$$\Phi_\lambda^{w,v} \overset{\text{def}}{=} (\Phi_{\lambda-\mathsf{wt}\, v}^{w} \otimes 1)\Phi_\lambda^v : M_\lambda \to M_{\lambda-\mathsf{wt}\, v-\mathsf{wt}\, w} \otimes W \otimes V.$$

The expectation value of this composition, $\langle \Phi_\lambda^{w,v} \rangle$, is a bilinear function of w and v. Therefore, there exists a linear operator $J_{WV}(\lambda) \in \mathsf{End}(W \otimes V)$ (of weight zero, i.e., commuting with $\mathfrak{h}$), such that $\langle \Phi_\lambda^{w,v} \rangle = J_{WV}(\lambda)(w \otimes v)$. In other words, we have $(\Phi_{\lambda-\mathsf{wt}\, v}^{w} \otimes 1)\Phi_\lambda^v = \Phi_\lambda^{J_{WV}(\lambda)(w\otimes v)}$. The operator $J_{WV}(\lambda)$ is called the *fusion operator* (because it tells us how to "fuse" two intertwining operators).

The fusion operator has a number of interesting properties, which we discuss below. In particular, it is lower triangular, i.e., has the form $J = 1 + N$, where N is a sum of terms which have strictly positive weights in the second component. Consequently, N is nilpotent, and J is invertible.

Define also the *exchange operator*,

$$R_{VW}(\lambda) \overset{\text{def}}{=} J_{VW}^{-1}(\lambda) J_{WV}^{21}(\lambda) : V \otimes W \to V \otimes W.$$

This operator tells us how to exchange the order of two intertwining operators, in the sense that if $R_{WV}(\lambda)(w \otimes v) = \sum_i w_i \otimes v_i$ (where w_i, v_i are homogeneous), then $\Phi_\lambda^{w,v} = P \sum_i \Phi_\lambda^{v_i, w_i}$ (where P permutes V and W).

1.2.3 Fusion and exchange operators for quantum groups

The fusion and exchange constructions generalize without significant changes to the case when the Lie algebra $\mathfrak{g}$ is replaced by the quantum group $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$, where q is not zero or a root of unity. The only change that needs to be made is in the definition of the exchange operator: namely, one sets $R(\lambda) = J_{VW}^{-1}(\lambda)\mathcal{R}^{21} J_{WV}^{21}(\lambda)$, where $\mathcal{R}$ is the universal R-matrix of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$. This is because when changing the order of intertwining operators, we must change the order of tensor product of representations $V \otimes W$, which in the quantum case is done by means of the R-matrix.

Theorem 1.19 (Etingof and Varchenko (1999)) *$R_{VV}(\lambda)$ is a solution of the quantum dynamical Yang–Baxter equation.*

Example 1.20 Let $\mathfrak{g} = \mathfrak{sl}_n$, and V be the vector representation of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$. Then the exchange operator has the form $R = \mathsf{q}^{1-1/n}\tilde{R}$, where $\tilde{R}$ is given by (1.1), with

$$\beta_{ab} = \frac{\mathsf{q}^{-2} - 1}{\mathsf{q}^{2(\lambda_a - \lambda_b - a + b)} - 1},$$
$$\alpha_{ab} = \mathsf{q}^{-1} \quad \text{if } a < b,$$

and

$$\alpha_{ab} = \frac{(\mathsf{q}^{2(\lambda_b-\lambda_a+a-b)} - \mathsf{q}^{-2})(\mathsf{q}^{2(\lambda_b-\lambda_a+a-b)} - \mathsf{q}^{2})}{\mathsf{q}(\mathsf{q}^{2(\lambda_b-\lambda_a+a-b)} - 1)^2} \quad \text{if } a > b.$$

The exchange operator for the vector representation of $\mathfrak{g}$ is obtained by passing to the limit $\mathsf{q} \to 1$; i.e., it is given by (1.1), with

$$\beta_{ab} = \frac{1}{\lambda_b - \lambda_a - b + a},$$
$$\alpha_{ab} = 1 \quad \text{for } a < b,$$

and

$$\alpha_{ab} = \frac{(\lambda_b - \lambda_a + a - b - 1)(\lambda_b - \lambda_a + a - b + 1)}{(\lambda_b - \lambda_a + a - b)^2} \quad \text{if } a > b.$$

It is easy to see that these exchange operators are gauge equivalent to the basic rational and trigonometric solutions of QDYBE, respectively.

1.2.4 The ABRR equation

The fusion operator is not only a tool to define the exchange operator satisfying QDYBE, but is an interesting object by itself, which deserves a separate study; so we will briefly discuss its properties.

Let ρ be the half-sum of positive roots of $\mathfrak{g}$. Let $\Theta(\lambda) \in \mathfrak{U}(\mathfrak{h})$ be given by $\Theta(\lambda) = \bar{\lambda} + \bar{\rho} - \frac{1}{2}\sum \mathsf{x}_i^2$. Then $\Theta(\lambda)$ defines an operator in any $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$-module with weight decomposition. Let $\mathcal{R}_0 = \mathcal{R}\mathsf{q}^{-\sum \mathsf{x}_i \otimes \mathsf{x}_i}$ be the unipotent part of the universal R-matrix.

Theorem 1.21 (ABRR equation, Arnaudon *et al.* (1998)) *For* $\mathsf{q} \neq 1$, *the fusion operator is a unique lower triangular zero weight operator, which satisfies the equation:*

$$J(\lambda)(1 \otimes \mathsf{q}^{2\Theta(\lambda)}) = \mathcal{R}_0^{21}(1 \otimes \mathsf{q}^{2\Theta(\lambda)})J(\lambda). \tag{1.4}$$

For $\mathsf{q} = 1$, *the fusion operator satisfies the classical limit of this equation:*

$$[J(\lambda), 1 \otimes \Theta(\lambda)] = (\sum_{\alpha>0} \mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha)J(\lambda), \tag{1.5}$$

(Here for brevity we have dropped the subscripts W and V, with the understanding that both sides are operators on $W \otimes V$.)

1.2.5 The universal fusion operator

Using the ABRR equation, we can define the universal fusion operator, living in a completion of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})^{\otimes 2}$, which becomes $J_{WV}(\lambda)$ after evaluating in $W \otimes V$. Namely, the universal fusion operator $J(\lambda)$ is the unique lower triangular solution of the ABRR equation in a completion of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})^{\otimes 2}$. This solution can be found in the form of a series $J = \sum_{n \geq 0} J_n$, $J_0 = 1$, where $J_n \in \mathfrak{U}_\mathsf{q}(\mathfrak{g}) \otimes \mathfrak{U}_\mathsf{q}(\mathfrak{g})$ has zero weight and its second component has degree n in principal gradation; so J_n are computed recursively.

This allows one to compute the universal fusion operator quite explicitly. For example, if $\mathsf{q} = 1$ and $\mathfrak{g} = \mathfrak{sl}_2$, then the universal fusion operator is given by the formula

$$J(\lambda) = \sum_{n\geq 0} \frac{(-1)^n}{n!} \mathsf{f}^n \otimes (\lambda - \mathsf{h} + n + 1)^{-1} \cdots (\lambda - \mathsf{h} + 2n)^{-1} \mathsf{e}^n.$$

1.2.6 The dynamical twist equation

Another important property of the fusion operator is the dynamical twist equation (which is a dynamical analog of the equation for a Drinfeld twist in a Hopf algebra).

Theorem 1.22 *The universal fusion operator $J(\lambda)$ satisfies the dynamical twist equation*

$$J^{12,3}(\lambda) J^{1,2}(\lambda - h^3) = J^{1,23}(\lambda) J^{2,3}(\lambda).$$

Here the superscripts of J stand for components on which the first and second components of J act; e.g., $J^{1,23}$ means $(1 \otimes \Delta)(J)$, and $J^{1,2}$ means $J \otimes 1$.

1.3 Traces of intertwining operators and Macdonald functions

In this section we discuss a connection between dynamical R-matrices and certain integrable systems and special functions (in particular, Macdonald functions). This connection is one of the important motivations for studying the theory of dynamical R-matrices.

1.3.1 Trace functions

Let V be a finite-dimensional representation of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ ($\mathsf{q} \neq 1$), such that $V[0] \neq 0$. Recall that for any $v \in V[0]$ and generic μ, one can define an intertwining operator Φ_μ^v such that $\langle \Phi_\mu^v \rangle = v$. Following Etingof and Varchenko (2000), set $\Psi^v(\lambda, \mu) = \mathsf{tr}|_{M_\mu} (\Phi_\mu^v \mathsf{q}^{2\bar{\lambda}})$. This is an infinite series in the variables $\mathsf{q}^{-\langle \lambda, \alpha_i \rangle}$ (where α_i are the simple roots) whose coefficients are rational functions of $\mathsf{q}^{\langle \mu, \alpha_i \rangle}$ (times a common factor $\mathsf{q}^{2\langle \lambda, \mu \rangle}$). For generic μ this series converges near 0, and its matrix elements belong to $\mathsf{q}^{2\langle \lambda, \mu \rangle}(\mathbf{C}(\mathsf{q}^{\langle \lambda, \alpha_i \rangle}) \otimes \mathbf{C}(\mathsf{q}^{\langle \mu, \alpha_i \rangle}))$.

Let $\Psi_V(\lambda, \mu)$ be the $\mathsf{End}(V[0])$-valued function with $\Psi_V(\lambda, \mu) v = \Psi^v(\lambda, \mu)$. The function Ψ_V has remarkable properties and in a special case is closely related to Macdonald functions. To formulate the properties of Ψ_V, we will consider a renormalized version of this function. Namely, let $\delta_{\mathsf{q}}(\lambda)$ be the Weyl denominator $\prod_{\alpha>0}(\mathsf{q}^{\langle \lambda, \alpha \rangle} - \mathsf{q}^{-\langle \lambda, \alpha \rangle})$. Let also $Q(\mu) = \sum S^{-1}(b_i) a_i$, where $\sum a_i \otimes b_i = J(\mu)$ is the universal fusion operator (this is an infinite expression, but it makes sense as a linear operator on finite-dimensional representations; moreover it is of zero weight and invertible). Define the *trace function*

$$F_V(\lambda, \mu) = \delta_{\mathsf{q}}(\lambda)\Psi_V(\lambda, -\mu - \rho)Q(-\mu - \rho)^{-1}.$$

1.3.2 Commuting difference operators

For any finite-dimensional $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$-module W, we define a difference operator $\mathbf{D}_W$ acting on functions on $\mathfrak{h}^*$ with values in $V[0]$. Namely, we set

$$(\mathbf{D}_W f)(\lambda) = \sum_{\nu \in \mathfrak{h}^*} \mathsf{tr}|_W \left(R_{WV}(-\lambda - \rho)\right) f(\lambda + \nu).$$

These operators are dynamical analogs of transfer matrices, and were introduced in Felder and Varchenko (1997). It can be shown that

$$\mathbf{D}_{W_1 \otimes W_2} = \mathbf{D}_{W_1} \mathbf{D}_{W_2};$$

in particular, $\mathbf{D}_W$ commute with each other, and the algebra generated by them is the polynomial algebra in $\mathbf{D}_{W_i}$, where W_i are the fundamental representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$.

1.3.3 Difference equations for the trace functions

It turns out that trace functions $F_V(\lambda, \mu)$, regarded as functions of λ, are common eigenfunctions of $\mathbf{D}_W$.

Theorem 1.23 (Etingof and Varchenko (2000)) *One has*

$$\mathbf{D}_W^{(\lambda)} F_V(\lambda, \mu) = \chi_W(\mathsf{q}^{-2\bar{\mu}}) F_V(\lambda, \mu), \tag{1.6}$$

where $\chi_W(x) = \mathsf{tr}|_W(x)$ *is the character of* W.

In fact, it is easy to deduce from this theorem that if v_i is a basis of $V[0]$ then $F_V(\lambda, \mu)v_i$ is a basis of solutions of (1.6) in the power series space. Thus, trace functions allow us to integrate the quantum integrable system defined by the commuting operators $\mathbf{D}_{W_i}$.

Theorem 1.24 (Etingof and Varchenko (2000)) *The function* F_V *is symmetric in* λ *and* μ *in the following sense:* $F_{V^*}(\mu, \lambda) = F_V(\lambda, \mu)^*$.

This symmetry property implies that F_V also satisfies "dual" difference equations with respect to μ: $\mathbf{D}_W^{(\mu)} F_V(\lambda, \mu)^* = \chi_W(q^{-2\bar{\lambda}}) F_V(\lambda, \mu)^*$.

1.3.4 Macdonald functions

An important special case of the theory of trace functions, worked out in Etingof and Kirillov (1994), is $\mathfrak{g} = \mathfrak{sl}_n$, and $V = L_{mn\omega_1}$, where ω_1 is the first fundamental weight, and m a non-negative integer. The zero-weight subspace of this representation is one-dimensional, so the function Ψ_V can be regarded as a scalar function. We will denote this scalar function by $\Psi_m(\mathsf{q},\lambda,\mu)$.

Recall the definition of Macdonald operators (Macdonald (1988), Etingof and Kirillov (1994)). They are operators on the space of functions $f(\lambda_1,...,\lambda_n)$ which are invariant under simultaneous shifting of the variables, $\lambda_i \to \lambda_i + c$, and have the form

$$M_r = \sum_{I\subset\{1,...,n\}:|I|=r}\left(\prod_{i\in I, j\notin I}\frac{\mathsf{t}\mathsf{q}^{2\lambda_i}-\mathsf{t}^{-1}\mathsf{q}^{2\lambda_j}}{\mathsf{q}^{2\lambda_i}-\mathsf{q}^{2\lambda_j}}\right)T_I,$$

where $T_I\lambda_j = \lambda_j$ if $j\notin I$ and $T_I\lambda_j=\lambda_j+1$ if $j\in I$. Here q,t are parameters. We will assume that $\mathsf{t}=\mathsf{q}^{m+1}$, where m is a nonnegative integer.

It is known (Macdonald (1988)) that the operators M_r commute. From this it can be deduced that for a generic $\mu=(\mu_1,...,\mu_n)$, $\sum\mu_i=0$, there exists a unique power series $f_{m0}(\mathsf{q},\lambda,\mu)\in \mathbf{C}[[\mathsf{q}^{\lambda_2-\lambda_1},...,\mathsf{q}^{\lambda_n-\lambda_{n-1}}]]$ such that the series $f_m(\mathsf{q},\lambda,\mu) \stackrel{\text{def}}{=} \mathsf{q}^{2\langle\lambda,\mu-m\rho\rangle}f_{m0}(\mathsf{q},\lambda,\mu)$ satisfies difference equations

$$M_r f_m(\mathsf{q},\lambda,\mu) = \left(\sum_{I\subset\{1,...,n\}:|I|=r}\mathsf{q}^{2\sum_{i\in I}(\mu+\rho)_i}\right)f_m(\mathsf{q},\lambda,\mu).$$

Remark 1.25 The series f_{m0} is convergent to an analytic (in fact, a trigonometric) function.

The following theorem is contained in Etingof and Kirillov (1994).

Theorem 1.26 *One has*

$$f_m(\mathsf{q},\lambda,\mu) = \gamma_m(\mathsf{q},\lambda)^{-1}\Psi_m(\mathsf{q}^{-1},-\lambda,\mu),$$

where

$$\gamma_m(\mathsf{q},\lambda) \stackrel{\text{def}}{=} \prod_{i=1}^{m}\prod_{l<j}(\mathsf{q}^{\lambda_l-\lambda_j}-\mathsf{q}^{2i}\mathsf{q}^{\lambda_j-\lambda_l}).$$

Let $\mathbf{D}_W(\mathsf{q}^{-1},-\lambda)$ denote the difference operator, obtained from the operator $\mathbf{D}_W$ defined above by the transformation $\mathsf{q}\to\mathsf{q}^{-1}$ and the change of coordinates $\lambda\to-\lambda$. Let $\Lambda^r\mathbf{C}^n$ denote the q-analog of the r^{th} fundamental representation of $\mathfrak{sl}_n$.

Theorem 1.27 (Felder and Varchenko (1997), Etingof and Varchenko (2000))

$$\mathbf{D}_{\Lambda^r\mathbf{C}^n}(\mathsf{q}^{-1},-\lambda) = \delta_{\mathsf{q}}(\lambda)\gamma_m(\mathsf{q},\lambda)\circ M_r\circ\gamma_m(\mathsf{q},\lambda)^{-1}\delta_{\mathsf{q}}(\lambda)^{-1}.$$

Remark 1.28 In the theory of trace functions, one may replace Verma modules M_μ with finite-dimensional irreducible modules L_μ with sufficiently large highest weight, and obtain results analogous to the above. In particular, one may set $\hat{\Psi}_m(\mathsf{q}, \lambda, \mu) = \mathsf{tr}(\hat{\Phi}_\mu^V \mathsf{q}^{2\bar{\lambda}})$, where $\hat{\Phi}_\mu^V : L_\mu \to L_\mu \otimes V \otimes V^*[0]$ is the intertwiner with highest coefficient 1 (such an operator exists if and only if $\mu - m\rho \geq 0$, see Etingof and Kirillov (1994)). Then one can show analogously to Theorem 1.26 (see Etingof and Kirillov (1994)) that the function

$$\hat{f}_m(\mathsf{q}, \lambda, \mu) \stackrel{\text{def}}{=} \gamma_m(\mathsf{q}, \lambda)^{-1} \hat{\Psi}_m(\mathsf{q}^{-1}, -\lambda, \mu + m\rho)$$

is the Macdonald polynomial $P_\mu(\mathsf{q}, \mathsf{t}, \mathsf{q}^{2\lambda})$ with highest weight μ (μ is a dominant integral weight). In this case, Theorem 1.23 says that Macdonald's polynomials are eigenfunctions of Macdonald's operators, Theorem 1.24 is the Macdonald symmetry identity (see Macdonald (1988)), and the dual version of Theorem 1.23 gives recursive relations for Macdonald's polynomials with respect to the weight (for $\mathfrak{sl}_2$—the usual 3-term relation for orthogonal polynomials).

Remark 1.29 If $\mathsf{q} = 1$, the difference equations of Theorem 1.23 become differential equations, which in the case $\mathfrak{g} = \mathfrak{sl}_n$, $V = L_{kn\omega_1}$ reduce to the trigonometric Calogero–Moser system. In this limit, the symmetry property is destroyed, but the "dual" difference equations remain valid, now with the exchange operator for $\mathfrak{g}$ rather than $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$. Thus, both for $\mathsf{q} = 1$ and $\mathsf{q} \neq 1$, common eigenfunctions satisfy additional difference equations with respect to eigenvalues—the so-called *bispectrality property*.

Remark 1.30 Apart from trace Ψ^v of a single intertwining operator multiplied by $\mathsf{q}^{2\bar{\lambda}}$, it is useful to consider the trace of a product of several such operators. After an appropriate renormalization, such multicomponent trace function (taking values in $\mathsf{End}((V_1 \otimes \cdots \otimes V_N)[0])$) satisfies multicomponent analogs of (1.6) and its dual version, as well as the symmetry. Furthermore, it satisfies an additional quantum Knizhnik–Zamolodchikov–Bernard equation, and its dual version (see Etingof and Varchenko (2000)).

Remark 1.31 The theory of trace functions can be generalized to the case of any generalized Belavin–Drinfeld triple; see Etingof and Schiffmann (2001*b*).

1.3.5 Dynamical Weyl groups

Trace functions $F_V(\lambda, \mu)$ are not Weyl group invariant. Rather, the diagonal action of the Weyl group multiplies them by certain operators, called the *dynamical Weyl group* operators. These operators were studied in Tarasov and Varchenko (2000), Etingof and Varchenko (2002) and play an important role in the theory of dynamical R-matrices and trace functions. So we conclude the introduction with a brief discussion of these operators.

Recall that a nonzero vector in a $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$-module is said to be singular if it is annihilated by Chevalley generators E_i for all i.

Let $\mathbf{W}$ be the Weyl group of $\mathfrak{g}$. Let $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition of $w \in \mathbf{W}$. Set $\alpha^l = \alpha_{i_l}$ and $\alpha^j = (s_{i_l} \dots s_{i_{j+1}})(\alpha_{i_j})$ for $j = 1, \dots, l-1$. For $\mu \in \mathfrak{h}^*$ let $n_j = 2\frac{\langle \mu+\rho, \alpha^j \rangle}{\langle \alpha^j, \alpha^j \rangle}$. For a dominant integral weight μ, the numbers n_j are positive integers. Let $d^j = d_{i_j}$ (where d_i are the symmetrizing numbers for the Cartan matrix). It is known that the collection of pairs of integers $(n_1, d^1), ..., (n_k, d^k)$ and the product $\mathsf{f}_{\alpha_{i_1}}^{n_1} \cdots \mathsf{f}_{\alpha_{i_l}}^{n_l}$ do not depend on the reduced decomposition.

Define a vector $x_{w \cdot \mu} \in M_\mu$ by

$$x_{w\cdot\mu} \;=\; \frac{\mathsf{f}_{\alpha_{i_1}}^{n_1}}{[n_1]_{\mathsf{q}^{d^1}}!} \cdots \frac{\mathsf{f}_{\alpha_{i_l}}^{n_l}}{[n_l]_{\mathsf{q}^{d^l}}!}\, x_\mu \,. \tag{1.7}$$

This vector is singular. It does not depend on the reduced decomposition.

Let V be a finite-dimensional $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$-module, and $w \in \mathbf{W}$. According to Tarasov and Varchenko (2000), Etingof and Varchenko (2002), there exists a unique operator $A_{w,V}(\mu) \in \mathsf{End}(V)$ which rationally depends on $\mathsf{q}^{2\langle \mu, \alpha_i \rangle}$ and has the following properties.

Let μ be a sufficiently large dominant integral weight. Let $u \in V[\nu]$ for some $\nu \in \mathfrak{h}^*$. Then

$$\Phi_\mu^u \, x_{w\cdot(\mu)} \;=\; x_{w\cdot(\mu-\nu)} \;\otimes\; A_{w,V}(\mu)\, u \;+\; \text{l.o.t.}, \tag{1.8}$$

where l.o.t. stands for "lower order terms", or terms of lower weight in the first component. The collection of operators $\{A_{w,V}(\mu)\}_{w\in\mathbf{W}}$ is called *the dynamical Weyl group*. Thus, the dynamical Weyl group describes restriction of intertwining operators to Verma submodules.

The operators of the dynamical Weyl group preserve the weight decomposition of V and satisfy the cocycle condition. Namely, if $w_1, w_2 \in \mathbf{W}$, $l(w_1 w_2) = l(w_1) + l(w_2)$ (where $l(w)$ is the length of w), then

$$A_{w_1 w_2, V}(\mu) \;=\; A_{w_1,V}(w_2 \cdot \mu)\; A_{w_2,V}(\mu)\,. \tag{1.9}$$

Moreover, according to Etingof and Varchenko (2002), on the subspace $V[0]$ this equation is satisfied without the assumption $l(w_1 w_2) = l(w_1) + l(w_2)$.

Finally, let us explain the connection between dynamical Weyl group and trace functions. Let $\mathcal{A}_{w,V}(\lambda) \overset{\text{def}}{=} A_{w,V}(-\lambda-\rho)$. Then the trace function $F^V(\lambda,\mu)$ has the following symmetry property with respect to the dynamical Weyl group (Etingof and Varchenko (2002)).

Theorem 1.32

$$F^V(\lambda,\mu) \;=\; \mathcal{A}_{w,V}(w^{-1}\lambda)\; F^V(w^{-1}\lambda, w^{-1}\mu)\; \mathcal{A}_{w,V^*}(w^{-1}\mu)^* \tag{1.10}$$

for any $w \in \mathbf{W}$.

This theorem plays an important role in the deeper theory of trace functions, which is developed in Etingof and Varchenko (2003). However, this is already outside the scope of this book.

2

BACKGROUND MATERIAL

This chapter contains well-known facts about semisimple finite-dimensional Lie algebras. For proofs the reader is referred to the standard text (Humphreys (1972)).

2.1 Facts about $\mathfrak{sl}_2$

$\mathfrak{sl}_2$ is a Lie algebra over $\mathbf{C}$, generated by three elements e, f and h such that

$$[\mathsf{h}, \mathsf{e}] = 2\mathsf{e}, [\mathsf{h}, \mathsf{f}] = -2\mathsf{f}, [\mathsf{e}, \mathsf{f}] = \mathsf{h}.$$

We have

$$\mathfrak{sl}_2 \cong \{A \in \mathsf{Mat}_2\,\mathbf{C}, \mathsf{tr}\,A = 0\},$$

where

$$[\mathsf{x}, \mathsf{y}] = \mathsf{xy} - \mathsf{yx}, \quad \mathsf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now let d be a positive integer, and let $V_d = \{a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d\}$, the set of homogeneous polynomials of degree d in x, y. $\mathfrak{sl}_2$ acts on V_d by $\mathsf{e} \mapsto x\frac{\partial}{\partial y}$, $\mathsf{f} \mapsto y\frac{\partial}{\partial x}$, $\mathsf{h} \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. h is diagonalizable in V_d, and we have

$$V_d = \bigoplus_{-d \le m \le d,\ m \equiv d \bmod 2} V_d[m],$$

where $V_d[m] = \{v \in V_d \text{ such that } \mathsf{h}v = mv\}$ (the weight subspace of weight m). The eigenvalues of h are $m = d, d-2, d-4, \ldots, -d$ with corresponding eigenvectors $x^d, x^{d-1}y, \ldots y^d$. The highest weight is d, the highest weight vector is $v_d = x^d$, so that a basis of V_d is $\{v_d, \mathsf{f}v_d, \mathsf{f}^2v_d, \ldots, \mathsf{f}^dv_d\}$, and $\mathsf{f}^{d+1}v_d = 0$.

Theorem 2.1

1. *V_d is irreducible;*
2. *Every irreducible finite-dimensional representation of $\mathfrak{sl}_2$ is isomorphic to V_d for some d;*
3. *Every finite-dimensional representation of $\mathfrak{sl}_2$ is isomorphic to a unique direct sum of the form $V \cong \oplus_d N_d V_d$.*

Corollary 2.2 *In any finite-dimensional representation of $\mathfrak{sl}_2$, h is diagonalizable.*

2.2 Semisimple finite-dimensional Lie algebras and roots

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{C}$. We say that $\mathfrak{g}$ is simple if every ideal of $\mathfrak{g}$ is 0 or $\mathfrak{g}$. We say that $\mathfrak{g}$ is semisimple if $\mathfrak{g}$ is a direct sum of simple Lie algebras.

Example 2.3 $\mathfrak{sl}_n = \{A \in \mathsf{Mat}_n \text{ such that } \mathsf{tr}\, A = 0\}$, with $[A, B] = AB - BA$, is simple.

In what follows, we will let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra.

Definition 2.4 *An element* $\mathsf{a} \in \mathfrak{g}$ *is semisimple if the operator* $\mathsf{ad}\,\mathsf{a} : \mathfrak{g} \to \mathfrak{g}$ *defined by* $\mathsf{ad}\,\mathsf{a}(\mathsf{x}) = [\mathsf{a}, \mathsf{x}]$ *is diagonalizable.*

Definition 2.5 *A Cartan subalgebra in* $\mathfrak{g}$ *is a maximal abelian Lie subalgebra in* $\mathfrak{g}$ *which consists of semisimple elements.*

Example 2.6 In $\mathfrak{sl}_2$, we have

$$\mathsf{ad}\,\mathsf{h} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

hence, $\langle \mathsf{h} \rangle$ is a Cartan subalgebra. However, $\mathsf{ad}\,\mathsf{e}$ is nilpotent, so e is not semisimple, and $\langle \mathsf{e} \rangle$ is not a Cartan subalgebra. In $\mathfrak{sl}_n$, the subalgebra $\mathfrak{h}$ of diagonal matrices is a Cartan subalgebra.

Fact 2.7 *If G is the Lie group corresponding to* $\mathfrak{g}$*, then G acts on* $\mathfrak{g}$ *by automorphisms, and all Cartan subalgebras are conjugate under this action.*

Now consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We have $\mathfrak{g} = \oplus_{\beta \in \mathfrak{h}^*} \mathfrak{g}[\beta]$, where $\mathfrak{g}[\beta] = \{\mathsf{a} \in \mathfrak{g}, [\mathsf{h}, \mathsf{a}] = \beta(\mathsf{h})\mathsf{a} \text{ for all } \mathsf{h} \in \mathfrak{h}\}$. Since $\mathfrak{g}[0] = \mathfrak{h}$, we can write

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\beta \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}[\beta] \right).$$

Proposition 2.8 $\dim \mathfrak{g}[\beta] \leq 1$ *if* $\beta \neq 0$.

Definition 2.9 $\beta \in \mathfrak{h}^* \setminus \{0\}$ *is called a root if* $\dim \mathfrak{g}[\beta] = 1$.

The set of roots is denoted by $\mathsf{R} \subset \mathfrak{h}^*$.

Proposition 2.10

1. $\alpha \in \mathsf{R} \Longrightarrow -\alpha \in \mathsf{R}$.

2. *$\mathbf{R}\mathsf{R} \subset \mathfrak{h}^*$ is a real form of $\mathfrak{h}^*$; i.e., $(\mathbf{R}\mathsf{R})_{\mathbf{C}} = \mathfrak{h}^*$.*

Now let $\mathfrak{h}_{\mathbf{R}} = (\mathbf{R}\mathsf{R})^*$, and pick $\mathsf{t} \in \mathfrak{h}_{\mathbf{R}}$ such that $\langle \mathsf{t}, \alpha \rangle \neq 0$ for all $\alpha \in \mathsf{R}$.

Definition 2.11 *A root $\alpha \in \mathsf{R}$ is positive ($\alpha > 0$) if $\langle \mathsf{t}, \alpha \rangle > 0$ and negative ($\alpha < 0$) if $\langle \mathsf{t}, \alpha \rangle < 0$.*

Let the set of positive roots be denoted by R_+ and the set of negative roots be denoted by R_-. Of course, $\mathsf{R} = \mathsf{R}_+ \cup \mathsf{R}_-$. We note that if $r = \mathsf{dim}\,\mathfrak{h} = \mathsf{rank}\,\mathfrak{g}$, then $\mathbf{R}\mathsf{R} \cong \mathbf{R}^r$.

Definition 2.12 *A maximal root is $\theta \in \mathsf{R}$ for which $\langle \mathsf{t}, \theta \rangle$ is maximal (such θ is unique).*

Definition 2.13 *$\alpha \in \mathsf{R}_+$ is simple if it cannot be nontrivially represented as a nonnegative integer linear combination of positive roots.*

The set of simple positive roots is denoted by $\Pi \subset \mathsf{R}_+$.

Proposition 2.14

1. *Π is a basis of $\mathfrak{h}^*$.*
2. *$\mathsf{R}_+ \subset \mathbf{Z}_+\Pi$.*

We will write $\Pi = \{\alpha_1, \ldots, \alpha_r\}$.

Example 2.15 Let us have a look at $\mathfrak{sl}_3$, with

$$\mathfrak{h} = \{A \in \mathfrak{sl}_3 \text{ diagonal with } \mathsf{tr}\, A = 0\}.$$

We have $\mathsf{dim}\,\mathfrak{h} = 2, \mathsf{dim}\,\mathfrak{g} = 8$ and thus $|\mathsf{R}| = 6$. The six roots, along with a possible choice of t and the corresponding simple roots, are illustrated in Fig. 2.1.

Example 2.16 In $\mathfrak{sl}_n$, we take $\mathfrak{h} = \{$diagonal traceless $n \times n$ matrices$\}$. We can write $\mathfrak{h}^* = \{(y_1, \ldots, y_n)$ such that $\sum_{i=1}^n y_i = 0\}$. The roots are $\alpha_{ij} = (\alpha_{ij}^{(1)}, \ldots, \alpha_{ij}^{(n)})$, where $\alpha_{ij}^{(k)}$ is 1 for $k = i$, -1 for $k = j$ and 0 otherwise. We take $\mathsf{t} = (n, n-1, \ldots, 1)$. Then $\alpha_{ij} > 0$ for $i < j$ and $\alpha_{ij} < 0$ for $i > j$. Since $\alpha_{ij} = \alpha_{ik} + \alpha_{kj}$ if $i < k < j$, we see that the simple roots are $\alpha_i = \alpha_{i,i+1}$. Each positive root can be written as a sum of simple roots as follows: $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$.

2.3 Inner product on a simple Lie algebra

Theorem 2.17 *Let $\mathfrak{g}$ be a simple Lie algebra. Then there exists a unique nonzero invariant inner product on $\mathfrak{g}$ (up to scaling).*

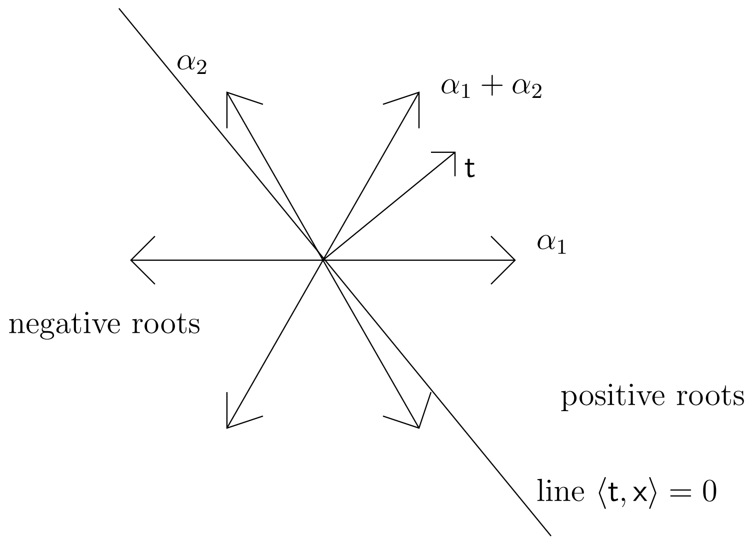

Fig. 2.1 Roots of $\mathfrak{sl}_3$.

Here invariant means that

$$\langle [\mathsf{x}, \mathsf{y}], \mathsf{z} \rangle + \langle \mathsf{y}, [\mathsf{x}, \mathsf{z}] \rangle = 0.$$

If $\mathsf{x} \in \mathfrak{g}[\beta]$ and $\mathsf{y} \in \mathfrak{g}[\alpha]$, then $\langle \mathsf{x}, \mathsf{y} \rangle = 0$ unless $\alpha + \beta = 0$. Furthermore, $\langle \cdot, \cdot \rangle|_{\mathfrak{h}}$ is nondegenerate and induces an inner product on $\mathfrak{h}^*$. Unless otherwise specified, we will normalize this inner product so that $\langle \theta, \theta \rangle = 2$. With this normalization, the inner product is positive definite on $\mathfrak{h}^*_{\mathbf{R}} = \mathbf{R}\mathsf{R}$.

2.4 Chevalley generators

Since $\mathfrak{g}_\beta = \mathfrak{g}[\beta]$ is always one-dimensional for $\beta \in \mathsf{R}$, we can write $\mathfrak{g}_{\alpha_i} = \mathsf{span}(\mathsf{e}_i)$ and $\mathfrak{g}_{-\alpha_i} = \mathsf{span}(\mathsf{f}_i)$, for $i = 1, \ldots, r$. We define $\mathsf{h}_i = [\mathsf{e}_i, \mathsf{f}_i] \in \mathfrak{h}$. Then $\mathsf{h}_1, \ldots, \mathsf{h}_r$ is a basis of $\mathfrak{h}$. Now $\alpha_i(\mathsf{h}_i) \neq 0$; we can thus normalize e_i and f_i so that $\alpha_i(\mathsf{h}_i) = 2$. We then have $[\mathsf{h}_i, \mathsf{e}_i] = \alpha_i(\mathsf{h}_i)\mathsf{e}_i = 2e_i$, $[\mathsf{h}_i, \mathsf{f}_i] = -\alpha_i(\mathsf{h}_i)\mathsf{f}_i = -2\mathsf{f}_i$. Therefore $(\mathfrak{sl}_2)_i = \langle \mathsf{e}_i, \mathsf{h}_i, \mathsf{f}_i \rangle$ is an sl_2-subalgebra of $\mathfrak{g}$.

Theorem 2.18 *The elements* $\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i, i = 1, \ldots, r$ *generate* $\mathfrak{g}$.

Definition 2.19 *The Cartan matrix of* $\mathfrak{g}$ *is the following* $r \times r$ *matrix:* $A = (a_{ij})$*, where* $a_{ij} = \alpha_j(\mathsf{h}_i)$.

Proposition 2.20 *The Cartan matrix has the following properties:*

1. $a_{ii} = 2$ *for all* $1 \leq i \leq r$*;*
2. $a_{ij} \in \mathbf{Z}$*;* $a_{ij} \leq 0$ *for all* $1 \leq i, j \leq r, i \neq j$*;*

3. $a_{ij} \neq 0 \iff a_{ji} \neq 0$;
4. *There exist unique relatively prime positive integers* d_i *such that* $d_i a_{ij} = d_j a_{ji}$;
5. *If* D *is the diagonal matrix with entries* d_i, *then* $\langle DAx, x\rangle > 0$ *for all* $x \neq 0$.

Furthermore, there exists a bijection between matrices satisfying 1–5 (modulo conjugation by a permutation matrix) and isomorphism classes of semisimple Lie algebras.

Theorem 2.21 *The defining relations for* $\mathsf{h}_i, \mathsf{e}_i, \mathsf{f}_i$ *are:*

1. $[\mathsf{h}_i, \mathsf{h}_j] = 0$ *for all* $1 \leq i, j \leq r$;
2. $[\mathsf{h}_i, \mathsf{e}_j] = a_{ij}\mathsf{e}_j$ *for all* $1 \leq i, j \leq r$;
3. $[\mathsf{h}_i, \mathsf{f}_j] = -a_{ij}\mathsf{f}_j$ *for all* $1 \leq i, j \leq r$;
4. $[\mathsf{e}_i, \mathsf{f}_j] = \delta_{ij}\mathsf{h}_i$ *for all* $1 \leq i, j \leq r$;
5. $(\mathsf{ad}\, \mathsf{e}_i)^{1-a_{ij}}\mathsf{e}_j = 0$ *for all* $1 \leq i, j \leq r, i \neq j$;
6. $(\mathsf{ad}\, \mathsf{f}_i)^{1-a_{ij}}\mathsf{f}_j = 0$ *for all* $1 \leq i, j \leq r, i \neq j$.

Relations 5 and 6 are called the *Serre relations*, and $\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i$ are known as the *Chevalley generators* of $\mathfrak{g}$.

2.5 Representations of finite-dimensional semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and V be a finite-dimensional representation of $\mathfrak{g}$.

Proposition 2.22 V *is diagonalizable under* $\mathfrak{h}$; *that is, we can write*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$$

where $V[\lambda] = \{v \in V, \mathsf{h}v = \lambda(\mathsf{h})v$ *for all* $\mathsf{h} \in \mathfrak{h}\}$.

$V[\lambda]$ is called the *weight subspace of* V *of weight* λ. We can put a partial order on $\mathfrak{h}^*$ as follows: $\mu \leq \lambda$ if $\mu = \lambda - \sum_{i=1}^{r} n_i\alpha_i$, where $n_i \in \mathbf{Z}_{\geq 0}$.

Definition 2.23 *The root lattice is* $\mathsf{Q} = \mathbf{Z}\Pi$.

We also write $\mathsf{Q}_+ = \mathbf{Z}_{\geq 0}\Pi$. Then $\mu \leq \lambda \iff \mu \in \lambda - \mathsf{Q}_+$.

Definition 2.24 *A weight of* V *is an element* $\lambda \in \mathfrak{h}^*$ *such that* $V[\lambda] \neq 0$. *A highest weight of* V *is a maximal element among weights of* V.

Proposition 2.25 *If* V *is irreducible, then the highest weight* $\lambda = \lambda(V)$ *is unique, and* $\mathsf{dim}\, V[\lambda] = 1$, *so* $V[\lambda] = \mathbf{C}\mathbf{v}_\lambda$, *where* $\mathbf{v}_\lambda$ *is a highest weight vector.*

Note that $\mathsf{h}_i\mathbf{v}_\lambda = \lambda(\mathsf{h}_i)\mathbf{v}_\lambda$; also, $\mathsf{e}_i\mathbf{v}_\lambda = 0$ since $\mathsf{wt}(\mathsf{e}_i\mathbf{v}_\lambda) = \alpha_i + \lambda > \lambda$, so $\alpha_i + \lambda$ is not a weight.

Theorem 2.26 *An irreducible finite-dimensional representation with a given highest weight λ, if it exists, is unique.*

Notation 2.27 *The representation in Theorem 2.26 is denoted by V_λ.*

Theorem 2.28 *Any finite-dimensional representation of $\mathfrak{g}$ is a direct sum of irreducible representations.*

Definition 2.29 *A weight $\lambda \in \mathfrak{h}^*$ is said to be integral if for all i, $\lambda(\mathsf{h}_i) \in \mathbf{Z}$. A weight $\lambda \in \mathfrak{h}^*$ is dominant integral if for all i, $\lambda(\mathsf{h}_i) \in \mathbf{Z}_{\geq 0}$.*

We write $\mathsf{P} = \{\text{integral weights}\}$ and $\mathsf{P}_+ = \{\text{dominant integral weights}\}$.

Theorem 2.30 *There exists a finite-dimensional irreducible representation V_λ with highest weight λ if and only if $\lambda \in \mathsf{P}_+$.*

Example 2.31 For $\mathfrak{sl}_2$, $\mathsf{P} = \mathbf{Z}$ and $\mathsf{P}_+ = \mathbf{Z}_{\geq 0}$.

We now ask ourselves the following question: how do we define highest weight representations for $\lambda \notin \mathsf{P}_+$?

Definition 2.32 *Given a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, we define $\mathfrak{n}_+$ to be the positive nilpotent subalgebra, generated by the $\mathsf{e}_i, 1 \leq i \leq r$. In other words, $\mathfrak{n}_+ = \oplus_{\alpha>0}\mathfrak{g}[\alpha]$. Similarly, we define $\mathfrak{n}_-$ to be the negative nilpotent subalgebra, generated by the $\mathsf{f}_i, 1 \leq i \leq r$. In other words, $\mathfrak{n}_- = \oplus_{\alpha<0}\mathfrak{g}[\alpha]$.*

Note that $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Definition 2.33 *The positive and negative Borel subalgebras are $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ and $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{h}$. In other words, $\mathfrak{b}_+$ is generated by the e_i and h_i and $\mathfrak{b}_-$ is generated by the f_i and h_i.*

We can regard $\lambda \in \mathfrak{h}^*$ as a one-dimensional representation of $\mathfrak{b}_+$, with $\lambda(\mathfrak{n}_+) = 0$.

Definition 2.34 *Given a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and a representation V of $\mathfrak{a}$, we define the induced representation of $\mathfrak{g}$: $\mathsf{Ind}_{\mathfrak{a}}^{\mathfrak{g}} V = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{a})} V$.*

Definition 2.35 *Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be semisimple, and $\lambda \in \mathfrak{h}^*$. The Verma module M_λ with highest weight λ is defined by*

$$M_\lambda = \mathsf{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \lambda.$$

Theorem 2.36 (follows from Poincaré–Birkhoff–Witt theorem) *The natural map $\mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{b}_+) \to \mathfrak{U}(\mathfrak{g})$ is an isomorphism.*

It follows that

$$\begin{aligned} M_\lambda &= \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \lambda \\ &= \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{b}_+) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \lambda \\ &= \mathfrak{U}(\mathfrak{n}_-) \otimes \lambda. \end{aligned}$$

We can thus write $M_\lambda = \mathfrak{U}(\mathfrak{n}_-)\mathbf{v}_\lambda$ (a free rank 1 $\mathfrak{U}(\mathfrak{n}_-)$-module with generator $\mathbf{v}_\lambda$), with $\mathsf{h}_i\mathbf{v}_\lambda = \lambda(\mathsf{h}_i)\mathbf{v}_\lambda$ and $\mathsf{e}_i\mathbf{v}_\lambda = 0$.

Corollary 2.37 (also follows from PBW) *For any ordering $\alpha^{(1)}, \ldots, \alpha^{(N)}$ of R_+, the elements $\mathsf{f}_{\alpha^{(1)}}^{n_1} \cdots \mathsf{f}_{\alpha^{(N)}}^{n_N}\mathbf{v}_\lambda$ form a basis of M_λ.*

Thus, the weights of M_λ all lie in $\lambda - \mathsf{Q}_+$.

Example 2.38 For $\mathfrak{sl}_2$, $M_\lambda = \mathsf{span}\{\mathbf{v}_\lambda, \mathsf{f}\mathbf{v}_\lambda, \mathsf{f}^2\mathbf{v}_\lambda, \ldots\}$, and for all $k \geq 0$, we have

$$\begin{aligned} \mathsf{h}\,(\mathsf{f}^k\mathbf{v}_\lambda) &= (\lambda - 2k)\mathsf{f}^k\mathbf{v}_\lambda \\ \mathsf{f}\,(\mathsf{f}^k\mathbf{v}_\lambda) &= \mathsf{f}^{k+1}\mathbf{v}_\lambda \\ \mathsf{e}\,(\mathsf{f}^k\mathbf{v}_\lambda) &= \mathsf{fef}^{k-1}\mathbf{v}_\lambda + \mathsf{hf}^{k-1}\mathbf{v}_\lambda \quad \text{since } [\mathsf{e},\mathsf{f}] = \mathsf{h} \\ &= \mathsf{fef}^{k-1}\mathbf{v}_\lambda + (\lambda - 2(k-1))\,\mathsf{f}^{k-1}\mathbf{v}_\lambda \\ &\quad\vdots \\ &= \lambda\mathsf{f}^{k-1}\mathbf{v}_\lambda + (\lambda-2)\mathsf{f}^{k-1}\mathbf{v}_\lambda + \cdots + (\lambda - 2(k-1))\,\mathsf{f}^{k-1}\mathbf{v}_\lambda \\ &= (k\lambda - k(k-1))\,\mathsf{f}^{k-1}\mathbf{v}_\lambda \\ &= k(\lambda - k + 1)\mathsf{f}^{k-1}\mathbf{v}_\lambda. \end{aligned}$$

2.6 Irreducible highest weight modules; the Shapovalov form

Let A be an algebra, and $B \subset A$ be a subalgebra. If M is a B-module, we can form an A-module $A \otimes_B M$ (the induced module).

Proposition 2.39 (Frobenius reciprocity) *If N is any A-module, then the natural map* $\mathsf{Hom}_A(A \otimes_B M, N) \to \mathsf{Hom}_B(M, N)$ *is an isomorphism.*

Corollary 2.40 *If N is any $\mathfrak{g}$-module, then* $\mathsf{Hom}_{\mathfrak{g}}(M_\lambda, N)$ *is canonically isomorphic to* $\mathsf{Hom}_{\mathfrak{b}_+}(\lambda, N) = \{v \in N, \mathsf{e}_i v = 0, \mathsf{h}v = \lambda(\mathsf{h})v,\ \mathsf{h} \in \mathfrak{h}\}$

Definition 2.41 *We let J_λ be the maximal proper submodule of M_λ.*

In other words, $J_\lambda = \sum_{N \subsetneq M_\lambda} N$. (This is indeed a proper submodule, since N has a weight decomposition, and $\mathbf{v}_\lambda$ is not in N for $N \subsetneq M_\lambda$.)

Definition 2.42 *We define $V_\lambda = M_\lambda / J_\lambda$, the irreducible $\mathfrak{g}$-module with highest weight λ.*

V_λ is finite-dimensional if and only if $\lambda \in \mathsf{P}_+$. We would like to know when M_λ is irreducible (in other words, when is $M_\lambda = V_\lambda$?)

Theorem 2.43 *There exists a unique bilinear form $\langle \cdot, \cdot \rangle$ on M_λ such that*

1. $\langle \mathbf{v}_\lambda, \mathbf{v}_\lambda \rangle = 1$;
2. $\langle \mathsf{e}_i v, w \rangle = \langle v, \mathsf{f}_i w \rangle$;
3. $\langle \mathsf{h}_i v, w \rangle = \langle v, \mathsf{h}_i w \rangle$;
4. $\langle \mathsf{f}_i v, w \rangle = \langle v, \mathsf{e}_i w \rangle$.

This bilinear form is symmetric.

Definition 2.44 *The symmetric bilinear form in theorem 2.43 is called the Shapovalov form.*

Proof Consider the automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ given by $\omega(\mathsf{e}_i) = \mathsf{f}_i$, $\omega(\mathsf{f}_i) = \mathsf{e}_i$, $\omega(\mathsf{h}_i) = -\mathsf{h}_i$ (the so-called Cartan involution). If N is a $\mathfrak{g}$-module, then we can define a new $\mathfrak{g}$-module N^ω as follows: N^ω is the same as N as a vector space, and $\rho_{N^\omega}(x) = \rho_N(\omega(x))$. There is a bijection between bilinear forms $\langle \cdot, \cdot \rangle : M_\lambda \otimes M_\lambda \to \mathbf{C}$ satisfying conditions 2, 3, 4, and linear maps $A : M_\lambda \to M_\lambda^*$ (where $M_\lambda^* = \oplus_\mu M_\lambda[\mu]^*$, the restricted dual of M_λ) defining a module homomorphism $M_\lambda \to M_\lambda^{\omega *}$. But $\mathsf{Hom}_{\mathfrak{g}}(M_\lambda, M_\lambda^{\omega *}) = \{v \in M_\lambda^{\omega *}$ such that $\mathsf{h}_i v = \lambda(\mathsf{h}_i)v, \mathsf{e}_i v = 0\} = \mathbf{C}\mathbf{v}_\lambda$. In order to satisfy condition 1, it suffices to normalize $A \in \mathsf{Hom}_{\mathfrak{g}}(M_\lambda, M_\lambda^{\omega *})$ so that $A\mathbf{v}_\lambda = \mathbf{v}_\lambda^*$, where $\langle \mathbf{v}_\lambda, \mathbf{v}_\lambda^* \rangle = 1$. This proves the existence and uniqueness of the form. To prove that it is symmetric, we need to show that $A^* = A$. Clearly, $A^* = \zeta A$ for some $\zeta \in \mathbf{C}$; thus $\langle v, w \rangle = \zeta \langle w, v \rangle$. Taking $v = w = \mathbf{v}_\lambda$ shows that $\zeta = 1$, and symmetry follows. □

We will now look at $\langle \cdot, \cdot \rangle$ on $M_\lambda[\lambda - \gamma]$, where $\gamma \in \mathsf{Q}_+$. Let $\mathsf{a}_1, \ldots, \mathsf{a}_l$ be a basis of $\mathfrak{U}(\mathfrak{n}_-)[-\gamma]$, and consider the matrix $B = (b_{ij}), b_{ij} = \langle \mathsf{a}_i \mathbf{v}_\lambda, \mathsf{a}_j \mathbf{v}_\lambda \rangle$. Let $F_\gamma(\lambda) = \mathsf{det} B$. As a function of λ, $F_\gamma(\lambda)$ is independent of the basis, up to scaling. Then $\{\lambda$ such that M_λ is not irreducible$\} = \cup_{\gamma \in \mathsf{Q}_+ \setminus \{0\}} \{\lambda$ such that $F_\gamma(\lambda) \neq 0\}$.

Definition 2.45 *The Kostant partition function is* $P : \mathsf{Q}_+ \to \mathbf{Z}_+$*, where* $P(\gamma)$ *is the number of representations of* γ *as a sum of positive roots. (Here representations that differ merely in the ordering of summands are regarded as identical.)*

Notation 2.46 *We will write*

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathsf{R}_+} \alpha \in \mathsf{P}_+.$$

Definition 2.47 *Suppose* $\{\mathsf{a}_i\}$ *is a basis of* $\mathfrak{g}$ *and* $\{\mathsf{a}^i\}$ *is the dual basis of* $\mathfrak{g}$ *(with respect to the invariant inner product). We define the Casimir element to be* $\mathsf{C} = \sum_i \mathsf{a}_i \mathsf{a}^i \in \mathfrak{U}(\mathfrak{g})$*.*

Proposition 2.48 *The Casimir element is central.*

Proposition 2.49 *For a semisimple Lie algebra* $\mathfrak{g}$ *and a highest weight module* V *with highest weight* λ*, the Casimir operator* $\mathsf{C} : V \to V$ *is equal to* $\langle \lambda, \lambda + 2\rho \rangle \, \mathsf{Id}$*.*

Proof We have $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \mathsf{R}} \mathfrak{g}_\alpha)$. Let e_α be a generator of $\mathfrak{g}_\alpha$. We know that if $\mathsf{a} \in \mathfrak{g}_\beta, \mathsf{b} \in \mathfrak{g}_\gamma$ and $\langle \mathsf{a}, \mathsf{b} \rangle \neq 0$, then $\beta + \gamma = 0$. Thus, $\langle \mathsf{e}_\alpha, \mathfrak{h} \rangle = 0$, $\langle \mathsf{e}_\alpha, \mathsf{e}_\beta \rangle = 0$ unless $\beta = -\alpha$, and we can normalize e_α so that $\langle \mathsf{e}_\alpha, \mathsf{e}_{-\alpha} \rangle = 1$. Now choose a basis x_i of $\mathfrak{h}$. Let x^i be its dual basis. Then the x_i and the e_α form a basis of $\mathfrak{g}$ and x^i and $\mathsf{e}_{-\alpha}$ form the dual basis. So the Casimir element is $\mathsf{C} = \sum_i \mathsf{x}_i \mathsf{x}^i + \sum_{\alpha \in \mathsf{R}} \mathsf{e}_\alpha \mathsf{e}_{-\alpha}$. Now

$$\begin{aligned}
\mathsf{C}\mathbf{v}_\lambda &= \sum_i \mathsf{x}_i \mathsf{x}^i \mathbf{v}_\lambda + \sum_{\alpha \in \mathsf{R}} \mathsf{e}_\alpha \mathsf{e}_{-\alpha} \mathbf{v}_\lambda \\
&= \sum_i \lambda(\mathsf{x}_i) \lambda(\mathsf{x}^i) \mathbf{v}_\lambda + \sum_{\alpha \in \mathsf{R}_+} \mathsf{e}_\alpha \mathsf{e}_{-\alpha} \mathbf{v}_\lambda \\
&= \langle \lambda, \lambda \rangle \mathbf{v}_\lambda + \sum_{\alpha \in \mathsf{R}_+} (\mathsf{e}_{-\alpha} \mathsf{e}_\alpha + \mathsf{h}_\alpha) \mathbf{v}_\lambda \quad \text{where } \mathsf{h}_\alpha = [\mathsf{e}_\alpha, \mathsf{e}_{-\alpha}] \in \mathfrak{h} \\
&= \langle \lambda, \lambda \rangle \mathbf{v}_\lambda + \lambda \left(\sum_{\alpha \in \mathsf{R}_+} \mathsf{h}_\alpha \right) \mathbf{v}_\lambda \\
&= \langle \lambda, \lambda \rangle \mathbf{v}_\lambda + \langle \lambda, 2\rho \rangle \mathbf{v}_\lambda \\
&= \langle \lambda, \lambda + 2\rho \rangle \mathbf{v}_\lambda .
\end{aligned}$$

The result follows since $V = \mathfrak{U}(\mathfrak{g}) \mathbf{v}_\lambda$ and C is central. □

For $\gamma \in \mathsf{Q}_+$, we will now consider $F_\gamma(\lambda)$, the determinant of the Shapovalov form on $M_\lambda[\lambda - \gamma]$.

Lemma 2.50 *$F_\gamma(\lambda)$ is a product of linear polynomials of the form*

$$\langle \lambda + \rho, \beta \rangle - \tfrac{1}{2} \langle \beta, \beta \rangle, \quad \beta \in \mathsf{Q}_+ \setminus \{0\}.$$

Proof If $F_{\gamma'}(\lambda) = 0$ for some γ', then M_λ is reducible. Let us take $J_\lambda \subset M_\lambda$, and consider any maximal weight of J_λ, say $\lambda - \beta, \beta \in \mathsf{Q}_+ \setminus \{0\}$. Let $u \in J_\lambda$ be a vector of this weight. Then $\mathsf{e}_i u = 0, \mathsf{h}_i u = (\lambda - \beta)(\mathsf{h}_i)u$. So $M_{\lambda-\beta} \cong \mathfrak{U}(\mathfrak{g})u \subset M_\lambda$; hence, $\mathsf{C}|_{M_\lambda} = \mathsf{C}|_{M_{\lambda-\beta}}$. Therefore,

$$\langle \lambda + 2\rho, \lambda \rangle = \langle \lambda - \beta + 2\rho, \lambda - \beta \rangle,$$

which simplifies to $\langle \lambda + \rho, \beta \rangle = \frac{1}{2} \langle \beta, \beta \rangle$. This completes the proof. □

Proposition 2.51 *If $\lambda \in \mathfrak{h}^*$ is such that $\langle \lambda + \rho, \alpha \rangle \neq \frac{n}{2} \langle \alpha, \alpha \rangle$ for all $\alpha \in \mathsf{R}_+$ and for all positive integers n, then M_λ is irreducible.*

Proof In view of Lemma 2.50, it remains to show that the elements γ that we get are multiples of roots. To see this, we will calculate the highest degree term of $F_\gamma(\lambda)$, and see that it is proportional to a product of $\langle \lambda, \alpha \rangle, \alpha \in \mathsf{R}_+$. Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ be the positive roots of $\mathfrak{g}$, and for $\mathbf{n} = (n_1, n_2, \ldots, n_N)$, write $\mathsf{f}^{\mathbf{n}} = \mathsf{f}^{n_1}_{\alpha^{(1)}} \mathsf{f}^{n_2}_{\alpha^{(2)}} \cdots \mathsf{f}^{n_N}_{\alpha^{(N)}}$. Let $a_{\mathbf{n},\mathbf{m}} = \langle \mathsf{f}^{\mathbf{n}} \mathbf{v}_\lambda, \mathsf{f}^{\mathbf{m}} \mathbf{v}_\lambda \rangle$. Then

$$\mathsf{det}(a_{\mathbf{n},\mathbf{m}}) = \sum_\sigma \mathsf{sign}\, \sigma \prod_{\mathbf{n}} a_{\mathbf{n},\sigma\mathbf{n}}.$$

It is easy to see that the term corresponding to $\sigma = \mathsf{Id}$ has a higher degree that the others. So the highest degree term is determined by the summand with $\sigma = \mathsf{Id}$ and hence equals $\prod_{\mathbf{n}} \prod_{\alpha \in \mathsf{R}_+} \langle \lambda, \alpha \rangle^{n_\alpha}$. The result follows. □

In fact, one could prove the following result:

Theorem 2.52 (see e.g. Kač and Kazhdan (1979)) *We have*

$$F_\gamma(\lambda) = c \prod_{\alpha \in \mathsf{R}_+} \prod_{n=1}^{\infty} \left(\langle \lambda + \rho, \alpha \rangle - \frac{n}{2} \langle \alpha, \alpha \rangle \right)^{P(\gamma - n\alpha)}.$$

Remark 2.53 The product in Theorem 2.52 is in fact finite since $P(\gamma - n\alpha) = 0$ for $n \gg 0$.

Lemma 2.50 and Proposition 2.51 imply the following corollary.

Corollary 2.54 *M_λ is irreducible for generic λ; more specifically, M_λ is irreducible if and only if for all positive integers n and $\alpha \in \mathsf{R}_+$, we have $\langle \lambda + \rho, \alpha \rangle \neq \frac{n}{2} \langle \alpha, \alpha \rangle$.*

Example 2.55 In $\mathfrak{sl}_2$, the only positive root is $\alpha = 2$, and thus $\rho = 1$. So M_λ will be irreducible if and only if $\langle \lambda + 1, 2 \rangle \neq \frac{n}{2} \langle 2, 2 \rangle$ for all positive integers n. Since we have $\langle x, y \rangle = \frac{1}{2}xy$, the above condition is equivalent to $\lambda + 1 \neq n$ for all positive integers n; therefore, M_λ is irreducible for $\lambda \neq 0, 1, 2, \ldots$.

3

INTERTWINERS, FUSION AND EXCHANGE OPERATORS FOR LIE ALGEBRAS

3.1 Intertwining operators

Definition 3.1 *Given two representations V_1 and V_2 of a Lie algebra $\mathfrak{g}$, we can define the tensor product $V_1 \otimes V_2$ to be the representation of $\mathfrak{g}$ given by the formula*

$$\pi_{V_1 \otimes V_2}(\mathsf{a}) = \pi_{V_1}(\mathsf{a}) \otimes 1 + 1 \otimes \pi_{V_2}(\mathsf{a}) \quad \text{for all } \mathsf{a} \in \mathfrak{g};$$

in other words, $\mathsf{a}(v_1 \otimes v_2) = \mathsf{a}v_1 \otimes v_2 + v_1 \otimes \mathsf{a}v_2$.

Definition 3.2 *Let V be a finite-dimensional representation of $\mathfrak{g}$. The expectation value map is the map $\langle\cdot\rangle : \mathsf{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu \otimes V) \to V[\lambda - \mu]$ given by*

$$\langle \Phi \rangle = \left\langle \mathbf{v}_\mu^*, \Phi \mathbf{v}_\lambda \right\rangle \stackrel{\text{def}}{=} (\mathbf{v}_\mu^* \otimes \mathsf{Id})(\Phi \mathbf{v}_\lambda),$$

where $\mathbf{v}_\mu^* \in M_\mu^*$ satisfies $\mathbf{v}_\mu^*(\mathbf{v}_\mu) = 1, \mathbf{v}_\mu^*(w) = 0$ for $\mathsf{wt}\, w < \mu$. ($\mathsf{wt}\, x$ denotes the weight of x.)

Thus $\Phi \mathbf{v}_\lambda = \mathbf{v}_\mu \otimes \langle \Phi \rangle + \sum_i a_i \otimes b_i$, where $\mathsf{wt}\, a_i < \mu, \mathsf{wt}\, b_i > \lambda - \mu$.

Lemma 3.3 *If M_μ is irreducible, then the expectation value map is an isomorphism of vector spaces.*

Proof In this proof, Y^* will denote the *restricted dual* of Y; i.e., the direct sum of the duals of the weight spaces.

$$\begin{aligned}
&\mathsf{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu \otimes V)\\
&\quad = \mathsf{Hom}_{\mathfrak{b}_+}(\lambda, M_\mu \otimes V) \quad \text{by Frobenius reciprocity}\\
&\quad = \mathsf{Hom}_{\mathfrak{b}_+}(\lambda \otimes M_\mu^*, V) \quad \text{since } \mathsf{Hom}(X, Y \otimes Z) = \mathsf{Hom}(Y^* \otimes X, Z)\\
&\quad = \mathsf{Hom}_{\mathfrak{b}_+}(M_\mu^*, (-\lambda) \otimes V)\\
&\quad = \mathsf{Hom}_{\mathfrak{b}_-}(M_\mu^{*\omega}, \lambda \otimes V^\omega) \quad \text{where } \omega \text{ is the Cartan involution.}
\end{aligned}$$

Since M_μ is irreducible, the Shapovalov form defines an injection $M_\mu \to M_\mu^{*\omega}$. This injection is actually an isomorphism (since the weight spaces of M_μ and $M_\mu^{*\omega}$ have the same dimension). Thus,

$$\begin{aligned}\mathsf{Hom}_{\mathfrak{b}_-}(M_\mu^{*\omega}, \lambda \otimes V^\omega) &= \mathsf{Hom}_{\mathfrak{b}_-}(M_\mu, \lambda \otimes V^\omega)\\ &= \mathsf{Hom}_{\mathfrak{h}}(\mu, \lambda \otimes V^\omega) \quad \text{by Frobenius reciprocity}\\ &= \mathsf{Hom}_{\mathfrak{h}}(\mu - \lambda, V^\omega)\\ &= \mathsf{Hom}_{\mathfrak{h}}(\lambda - \mu, V)\\ &= V[\lambda - \mu].\end{aligned}$$

□

Corollary 3.4 *If M_μ is irreducible, then for all homogeneous $v \in V$, there exists a unique intertwining operator $\Phi : M_{\mu+\mathsf{wt}\, v} \to M_\mu \otimes V$ such that $\langle \Phi \rangle = v$.*

Notation 3.5 *The intertwining operator Φ in Corollary 3.4 will be denoted by $\Phi^v_{\mu+\mathsf{wt}\, v}$.*

3.2 The fusion operator

Let V, W be finite-dimensional representations of $\mathfrak{g}$. Fix a generic λ. Let $\gamma, \beta \in \mathfrak{h}^*$, and $w \in W[\gamma], v \in V[\beta]$. The assignment

$$w, v \to \left\langle (\Phi^w_{\lambda-\beta} \otimes \mathsf{Id})\Phi^v_\lambda \right\rangle \in (W \otimes V)[\gamma + \beta]$$

is bilinear, so it extends to a linear map $W[\gamma]\otimes V[\beta] \to (W\otimes V)[\gamma+\beta]$. Combining these maps for all γ, β, we get a linear map $J_{WV}(\lambda) : W \otimes V \to W \otimes V$.

Definition 3.6 *The $J_{WV}(\lambda)$ that we just described is called the fusion operator or fusion matrix.*

Proposition 3.7

1. *$J_{WV}(\lambda)$ has zero weight: for any weight δ, $J_{WV}(\lambda)$ maps $(W \otimes V)[\delta]$ into itself.*
2. *$J_{WV}(\lambda)$ is lower triangular with respect to the weight decomposition, and has ones on its diagonal. That is, $J_{WV}(\lambda)(w \otimes v) = w \otimes v + \sum_i c_i \otimes b_i$, where $\mathsf{wt}\, c_i < \mathsf{wt}\, w, \mathsf{wt}\, b_i > \mathsf{wt}\, v$, for all homogeneous v, w. In particular, $J_{WV}(\lambda)$ is invertible whenever defined.*
3. *$J_{WV}(\lambda)$ is a rational function of λ.*

Proof Properties 1 and 3 are obvious.

Proof of Property 2. We have

$$\Phi^v_\lambda \mathbf{v}_\lambda = \mathbf{v}_{\lambda-\mathsf{wt}\, v} \otimes v + \sum a_i \otimes b_i,$$

where $\mathsf{wt}\, a_i < \lambda - \mathsf{wt}\, v$ and $\mathsf{wt}\, b_i > \mathsf{wt}\, v$. We now apply $\Phi^w_{\lambda-\mathsf{wt}\, v} \otimes \mathsf{Id}$ and get

$$(\Phi^w_{\lambda-\mathsf{wt}\, v} \otimes 1)\Phi^v_\lambda \mathbf{v}_\lambda = \mathbf{v}_{\lambda-\mathsf{wt}\, v-\mathsf{wt}\, w} \otimes w \otimes v + \sum \Phi^w_{\lambda-\mathsf{wt}(v)} a_i \otimes b_i + \text{l.w.t.},$$

where "l.w.t." denotes lower weight terms in the first component.

Thus,

$$\mathbf{v}^*_{\lambda-\mathsf{wt}\,v-\mathsf{wt}\,w}((\Phi^w_{\lambda-\mathsf{wt}\,v}\otimes 1)\Phi^v_\lambda \mathbf{v}_\lambda) = w\otimes v + \sum_i c_i\otimes b_i,$$

where $\mathsf{wt}\,c_i < \mathsf{wt}\,w, \mathsf{wt}\,b_i > \mathsf{wt}\,v$, and this completes the proof. □

3.3 The dynamical twist equation

Let V, W, U be finite-dimensional representations of the Lie algebra $\mathfrak{g}$.

Theorem 3.8 *Fusion operators satisfy the following dynamical twist (2-cocycle) equation in $V\otimes W\otimes U$:*

$$J^{12,3}_{V\otimes W,U}(\lambda)J^{12}_{VW}(\lambda-h^3) = J^{1,23}_{V,W\otimes U}(\lambda)J^{23}_{WU}(\lambda),$$

where $J^{12}_{VW}(\lambda-h^3)(v\otimes w\otimes u) \stackrel{\text{def}}{=} (J_{VW}(\lambda-\mathsf{wt}\,u)(v\otimes w))\otimes u$ ("dynamical notation").

Proof We first note that since $J_{VW}(\lambda)(v\otimes w) = \langle(\Phi^v_{\lambda-\mathsf{wt}\,w}\otimes\mathsf{Id})\Phi^w_\lambda\rangle$, we have

$$\Phi^{J_{VW}(\lambda)(v\otimes w)}_\lambda = (\Phi^v_{\lambda-\mathsf{wt}\,w}\otimes\mathsf{Id})\Phi^w_\lambda.$$

We will now compute the threefold composition

$$(\Phi^v_{\lambda-\mathsf{wt}\,u-\mathsf{wt}\,w}\otimes\mathsf{Id}\otimes\mathsf{Id})(\Phi^w_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id})\Phi^u_\lambda : M_\lambda\to M_{\lambda-\mathsf{wt}\,u-\mathsf{wt}\,v-\mathsf{wt}\,w}\otimes V\otimes W\otimes U$$

in two different ways. On the one hand,

$$(\Phi^w_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id})\Phi^u_\lambda = \Phi^{J_{WU}(\lambda)(w\otimes u)}_\lambda$$

and thus

$$\begin{aligned}&(\Phi^v_{\lambda-\mathsf{wt}\,u-\mathsf{wt}\,w}\otimes\mathsf{Id}\otimes\mathsf{Id})\left((\Phi^w_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id})\Phi^u_\lambda\right)\\ &\quad= (\Phi^v_{\lambda-\mathsf{wt}\,u-\mathsf{wt}\,w}\otimes\mathsf{Id}\otimes\mathsf{Id})\Phi^{J_{WU}(\lambda)(w\otimes u)}_\lambda\\ &\quad= \Phi^{J_{V,W\otimes U}(\lambda)(v\otimes J_{WU}(\lambda)(w\otimes u))}_\lambda;\end{aligned}$$

on the other hand,

$$(\Phi^v_{\lambda-\mathsf{wt}\,u-\mathsf{wt}\,w}\otimes\mathsf{Id}\otimes\mathsf{Id})(\Phi^w_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id}) = \Phi^{J_{VW}(\lambda-\mathsf{wt}\,u)(v\otimes w)}_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id}$$

and hence

$$\begin{aligned}&\left((\Phi^v_{\lambda-\mathsf{wt}\,u-\mathsf{wt}\,w}\otimes\mathsf{Id}\otimes\mathsf{Id})(\Phi^w_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id})\right)\Phi^u_\lambda\\ &\quad= (\Phi^{J_{VW}(\lambda-\mathsf{wt}\,u)(v\otimes w)}_{\lambda-\mathsf{wt}\,u}\otimes\mathsf{Id})\Phi^u_\lambda\\ &\quad= \Phi^{J_{V\otimes W,U}(\lambda)(J_{VW}(\lambda-\mathsf{wt}\,u)(v\otimes w)\otimes u)}_\lambda.\end{aligned}$$

Thus, we get

$$\Phi_\lambda^{J_{V,W\otimes U}(\lambda)(v\otimes J_{WU}(\lambda)(w\otimes u))} = \Phi_\lambda^{J_{V\otimes W,U}(\lambda)(J_{VW}(\lambda-\mathsf{wt}\,u)(v\otimes w)\otimes u)};$$

applying $\langle\cdot\rangle$ to both sides gives

$$J_{V,W\otimes U}(\lambda)\left(v\otimes J_{WU}(w\otimes u)\right) = J_{V\otimes W,U}(\lambda)\left(J_{VW}(\lambda-\mathsf{wt}\,u)(v\otimes w)\otimes u\right),$$

for all $v\in V, w\in W, u\in U$. Therefore,

$$J^{1,23}_{V,W\otimes U}(\lambda)J^{23}_{WU}(\lambda) = J^{12,3}_{V\otimes W,U}(\lambda)J^{12}_{VW}(\lambda-h^3),$$

as required. □

Notation 3.9 *Let $P_{12} : W\otimes V \to V\otimes W$ denote the permutation operator, and let $J^{21}_{VW}(\lambda) : W\otimes V \to W\otimes V$ be defined as follows: $J^{21}_{VW} \overset{\text{def}}{=} P_{12}J_{VW}(\lambda)P_{12}$. We can then define the following operators on $V\otimes W\otimes U$:*

$$J^{13}_{VU}(\lambda-h^2) \overset{\text{def}}{=} P_{23}J^{12}_{VU}(\lambda-h^3)P_{23} \quad \textit{and}$$
$$J^{23}_{WU}(\lambda-h^1) \overset{\text{def}}{=} P_{12}P_{23}J^{12}_{WU}(\lambda-h^3)P_{23}P_{12}.$$

Other fusion operators can be defined in the obvious way; e.g.,

$$J^{13,2}_{V\otimes U,W}(\lambda) \overset{\text{def}}{=} P_{23}J^{12,3}_{V\otimes U,W}P_{23}.$$

Corollary 3.10

$$J^{13,2}_{V\otimes U,W}(\lambda)J^{13}_{VU}(\lambda-h^2) = J^{1,23}_{V,W\otimes U}(\lambda)J^{32}_{UW}(\lambda).$$

Corollary 3.11

$$J^{23,1}_{W\otimes U,V}(\lambda)J^{23}_{WU}(\lambda-h^1) = J^{2,13}_{W,V\otimes U}(\lambda)J^{31}_{UV}(\lambda).$$

3.4 The exchange operator

Definition 3.12 *An R-matrix on a vector space V is an invertible linear operator $R : V\otimes V\to V\otimes V$ such that on $V\otimes V\otimes V$ we have*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}. \tag{3.1}$$

Equation (3.1) is known as the quantum Yang–Baxter equation. In this section, we will define the exchange operator and prove that it satisfies a generalization of Equation (3.1), known as the *quantum dynamical Yang–Baxter equation.*

Definition 3.13 *The exchange operator $R_{VW}(\lambda) : V \otimes W \to V \otimes W$ is defined by the formula*

$$R_{VW}(\lambda) = J_{VW}(\lambda)^{-1} J^{21}_{WV}(\lambda).$$

Proposition 3.14 *Suppose v, w are homogeneous vectors such that*

$$R_{VW}(\lambda)(v \otimes w) = \sum_i v_i \otimes w_i.$$

Then,

$$(\Phi^w_{\lambda - \mathsf{wt}\, v} \otimes \mathsf{Id})\Phi^v_\lambda = \sum_i P_{12}(\Phi^{v_i}_{\lambda - \mathsf{wt}\, w_i} \otimes \mathsf{Id})\Phi^{w_i}_\lambda P_{12}.$$

Proof The left-hand side is

$$(\Phi^w_{\lambda - \mathsf{wt}\, v} \otimes \mathsf{Id})\Phi^v_\lambda = \Phi^{J_{WV}(\lambda)(w \otimes v)}_\lambda.$$

The right-hand side is

$$\begin{aligned}\sum_i P_{12}(\Phi^{v_i}_{\lambda - \mathsf{wt}\, w_i} \otimes \mathsf{Id})\Phi^{w_i}_\lambda P_{12} &= P_{12} \sum_i \left(\Phi^{J_{VW}(\lambda)(v_i \otimes w_i)}_\lambda\right) P_{12} \\ &= P_{12} \Phi^{J_{VW}(\lambda) R_{VW}(\lambda)(v \otimes w)}_\lambda P_{12} \\ &= \Phi^{P_{12} J_{VW}(\lambda) R_{VW}(\lambda) P_{12}(w \otimes v)}_\lambda \\ &= \Phi^{J_{WV}(\lambda)(w \otimes v)}_\lambda\end{aligned}$$

Thus the two sides are equal. □

Theorem 3.15 (quantum dynamical Yang–Baxter equation) *Let V, W, U be finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. The exchange operator satisfies the following equation in $V \otimes W \otimes U$:*

$$R^{12}_{VW}(\lambda - h^3) R^{13}_{VU}(\lambda) R^{23}_{WU}(\lambda - h^1) = R^{23}_{WU}(\lambda) R^{13}_{VU}(\lambda - h^2) R^{12}_{VW}(\lambda). \qquad (3.2)$$

First proof, using the dynamical twist equation On the one hand, we have

$$\begin{aligned}&R^{12}_{VW}(\lambda-h^3)R^{13}_{VU}(\lambda)R^{23}_{WU}(\lambda-h^1)\\&\quad=J^{12}_{VW}(\lambda-h^3)^{-1}J^{21}_{WV}(\lambda-h^3)J^{13}_{VU}(\lambda)^{-1}\\&\qquad J^{31}_{UV}(\lambda)J^{23}_{WU}(\lambda-h^1)^{-1}J^{32}_{UW}(\lambda-h^1),\end{aligned}\tag{3.3}$$

by the definition of the exchange operator. Using Theorem 3.8 and Corollary 3.11, we see that

$$J^{12}_{VW}(\lambda-h^3)^{-1}=J^{23}_{WU}(\lambda)^{-1}J^{1,23}_{V,W\otimes U}(\lambda)^{-1}J^{12,3}_{V\otimes W,U}(\lambda),\tag{3.4}$$

$$J^{21}_{WV}(\lambda-h^3)=J^{12,3}_{V\otimes W,U}(\lambda)^{-1}J^{2,13}_{W,V\otimes U}(\lambda)J^{13}_{VU}(\lambda),\tag{3.5}$$

$$J^{23}_{WU}(\lambda-h^1)^{-1}=J^{31}_{UV}(\lambda)^{-1},J^{2,13}_{W,V\otimes U}(\lambda)^{-1}J^{23,1}_{W\otimes U,V}(\lambda)\tag{3.6}$$

and

$$J^{32}_{UW}(\lambda-h^1)=J^{23,1}_{W\otimes U,V}(\lambda)^{-1}J^{3,12}_{U,V\otimes W}(\lambda)J^{21}_{WV}(\lambda).\tag{3.7}$$

Substituting (3.4), (3.5), (3.6) and (3.7) into (3.3), we obtain

$$R^{12}_{VW}(\lambda-h^3)R^{13}_{VU}(\lambda)R^{23}_{WU}(\lambda-h^1)=J^{23}_{WU}(\lambda)^{-1}J^{1,23}_{V,W\otimes U}(\lambda)^{-1}J^{3,12}_{U,V\otimes W}(\lambda)J^{21}_{WV}(\lambda).\tag{3.8}$$

On the other hand, we have

$$\begin{aligned}&R^{23}_{WU}(\lambda)R^{13}_{VU}(\lambda-h^2)R^{12}_{VW}(\lambda)\\&\quad=J^{23}_{WU}(\lambda)^{-1}J^{32}_{UW}(\lambda)J^{13}_{VU}(\lambda-h^2)^{-1}J^{31}_{UV}(\lambda-h^2)J^{12}_{VW}(\lambda)^{-1}J^{21}_{WV}(\lambda).\end{aligned}\tag{3.9}$$

Using Corollary 3.10, we see that

$$J^{13}_{VU}(\lambda-h^2)^{-1}=J^{32}_{UW}(\lambda)^{-1}J^{1,23}_{V,W\otimes U}(\lambda)^{-1}J^{13,2}_{V\otimes U,W}(\lambda)\tag{3.10}$$

and

$$J^{31}_{UV}(\lambda-h^2)=J^{13,2}_{V\otimes U,W}(\lambda)^{-1}J^{3,12}_{U,V\otimes W}(\lambda)J^{12}_{VW}(\lambda).\tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we obtain

$$R^{23}_{WU}(\lambda)R^{13}_{VU}(\lambda-h^2)R^{12}_{VW}(\lambda)=J^{23}_{WU}(\lambda)^{-1}J^{1,23}_{V,W\otimes U}(\lambda)^{-1}J^{3,12}_{U,V\otimes W}(\lambda)J^{21}_{WV}(\lambda).\tag{3.12}$$

Comparing (3.8) and (3.12), we find that

$$R^{12}_{VW}(\lambda-h^3)R^{13}_{VU}(\lambda)R^{23}_{WU}(\lambda-h^1)=R^{23}_{WU}(\lambda)R^{13}_{VU}(\lambda-h^2)R^{12}_{VW}(\lambda),$$

as required. □

Second proof We will first introduce new notation. Suppose $V_1, \dots, V_n$ are representations of $\mathfrak{g}$. For $z \in (V_1 \otimes \cdots \otimes V_n)[\beta]$, we will define $\Phi^z_{\lambda,n} : M_\lambda \to M_{\lambda-\beta} \otimes V_1 \otimes \cdots \otimes V_n$ so that

$$\Phi^{v_1\otimes\cdots\otimes v_n}_{\lambda,n} = (\Phi^{v_1}_{\lambda-\mathsf{wt}\,v_n-\cdots-\mathsf{wt}\,v_2} \otimes \mathsf{Id} \otimes \cdots \otimes \mathsf{Id}) \cdots (\Phi^{v_{n-1}}_{\lambda-\mathsf{wt}\,v_n} \otimes \mathsf{Id})\Phi^{v_n}_\lambda,$$

whenever $v_i \in V_i$ are homogeneous vectors such that $\sum_i \mathsf{wt}\,v_i = \beta$, and $\Phi^{z_1+z_2}_{\lambda,n} = \Phi^{z_1}_{\lambda,n} + \Phi^{z_2}_{\lambda,n}$ for all $z_1, z_2 \in V_1 \otimes \cdots \otimes V_n$.

For example, $\Phi^z_{\lambda,2} = \Phi^{J_{V_1V_2}(\lambda)z}_\lambda$. Thus, Proposition 3.14 can be restated as

$$\Phi^{w\otimes v}_{\lambda,2} = P_{12}\Phi^{R^{12}(\lambda)P_{12}(w\otimes v)}_{\lambda,2}.$$

Thus we have, on the one hand:

$$\begin{aligned}
\Phi^{v_1\otimes v_2\otimes v_3}_{\lambda,3} &= P_{23}\Phi^{R^{23}(\lambda)P_{23}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{23}P_{12}\Phi^{R^{12}(\lambda-h^3)P_{12}R^{23}(\lambda)P_{23}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{23}P_{12}P_{23}\Phi^{R^{23}(\lambda)P_{23}R^{12}(\lambda-h^3)P_{12}R^{23}(\lambda)P_{23}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{23}P_{12}P_{23}\Phi^{R^{23}(\lambda)R^{13}(\lambda-h^2)R^{12}(\lambda)P_{23}P_{12}P_{23}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{13}\Phi^{R^{23}(\lambda)R^{13}(\lambda-h^2)R^{12}(\lambda)P_{13}(v_1\otimes v_2\otimes v_3)}_{\lambda,3}.
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
\Phi^{v_1\otimes v_2\otimes v_3}_{\lambda,3} &= P_{12}\Phi^{R^{12}(\lambda-h^3)P_{12}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{12}P_{23}\Phi^{R^{23}(\lambda)P_{23}R^{12}(\lambda-h^3)P_{12}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{12}P_{23}P_{12}\Phi^{R^{12}(\lambda-h^3)P_{12}R^{23}(\lambda)P_{23}R^{12}(\lambda-h^3)P_{12}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{12}P_{23}P_{12}\Phi^{R^{12}(\lambda-h^3)R^{13}(\lambda)R^{23}(\lambda-h^1)P_{12}P_{23}P_{12}(v_1\otimes v_2\otimes v_3)}_{\lambda,3} \\
&= P_{13}\Phi^{R^{23}(\lambda)R^{13}(\lambda-h^2)R^{12}(\lambda)P_{13}(v_1\otimes v_2\otimes v_3)}_{\lambda,3}.
\end{aligned}$$

Comparing the two, we obtain

$$R^{23}(\lambda)R^{13}(\lambda-h^2)R^{12}(\lambda)P_{13} = R^{23}(\lambda)R^{13}(\lambda-h^2)R^{12}(\lambda)P_{13},$$

and the quantum dynamical Yang–Baxter equation follows. □

It should be noted that the first proof, unlike the second, relies only on the fact that fusion operators satisfy the dynamical twist equation. Hence, it can be generalized to any situation when this equation is satisfied.

Definition 3.16 *Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra, and $V = \oplus_{\mu\in\mathfrak{h}^*}V[\mu]$ a diagonalizable finite-dimensional $\mathfrak{h}$-module. A quantum dynamical R-matrix is a meromorphic function*

$$R : \mathfrak{h}^* \to \mathsf{End}_{\mathfrak{h}}(V \otimes V)$$

which satisfies (3.2), *the quantum dynamical Yang–Baxter equation.*

Example 3.17 Let $\mathfrak{g} = \mathfrak{sl}_2$; then $\mathfrak{h}^* = \mathbf{C}$. Take V to be the two-dimensional irreducible representation of $\mathfrak{g}$; we may write $V = \mathbf{C}v_+ \oplus \mathbf{C}v_-$. We then have

$$\mathsf{h}v_+ = v_+, \mathsf{h}v_- = -v_-, \mathsf{e}v_+ = 0, \mathsf{e}v_- = v_+, \mathsf{f}v_+ = v_-, \mathsf{f}v_- = 0.$$

We would like to compute $J_{VV}(\lambda)$ and $R_{VV}(\lambda)$.

We note that $V \otimes V$ has a basis consisting of four elements:

- $v_+ \otimes v_+$ (of weight 2),
- $v_+ \otimes v_-$ (of weight 0),
- $v_- \otimes v_+$ (of weight 0),
- $v_- \otimes v_-$ (of weight -2).

Now we know that the fusion operator is lower triangular with ones on the diagonal; hence, each of these four basis elements must be fixed by $J_{VV}(\lambda)$, except $v_+ \otimes v_-$.

So we must compute $J_{VV}(\lambda)(v_+ \otimes v_-)$. We have

$$\Phi_\lambda^{v_-} \mathbf{v}_\lambda = \mathbf{v}_{\lambda+1} \otimes v_- + a(\lambda)\mathsf{f}\mathbf{v}_{\lambda+1} \otimes v_+,$$

for some function a of λ. We then note that

$$\left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, \Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle = v_+,$$

and

$$\begin{aligned}
&\left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, \Phi_{\lambda+1}^{v_+} \mathsf{f}\mathbf{v}_{\lambda+1} \right\rangle \\
&\quad = \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, (\mathsf{f} \otimes \mathsf{Id} + \mathsf{Id} \otimes \mathsf{f})\Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle \\
&\quad = \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, (\mathsf{Id} \otimes \mathsf{f})\Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle \quad \text{since } \langle \mathbf{v}_\lambda^*, \mathsf{f}w \rangle = 0 \text{ for all } w \in M_\lambda \\
&\quad = \mathsf{f} \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, \Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle \\
&\quad = \mathsf{f}v_+ = v_-.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_{VV}(\lambda)(v_+ \otimes v_-) &= \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id} \otimes \mathsf{Id}, (\Phi_{\lambda+1}^{v_+} \otimes \mathsf{Id})\Phi_\lambda^{v_-} \mathbf{v}_\lambda \right\rangle \\
&= \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id} \otimes \mathsf{Id}, (\Phi_{\lambda+1}^{v_+} \otimes \mathsf{Id})(\mathbf{v}_{\lambda+1} \otimes v_- + a(\lambda)\mathsf{f}\mathbf{v}_{\lambda+1} \otimes v_+) \right\rangle \\
&= v_+ \otimes v_- + a(\lambda) v_- \otimes v_+.
\end{aligned}$$

Finally, we must determine $a(\lambda)$. We see that

$$\begin{aligned}
0 &= \Phi_\lambda^{v_-} \mathsf{e}\mathbf{v}_\lambda \\
&= \mathsf{e}\Phi_\lambda^{v_-} \mathbf{v}_\lambda \\
&= \mathsf{e}(\mathbf{v}_{\lambda+1} \otimes v_-) + a(\lambda)\mathsf{e}(\mathsf{f}\mathbf{v}_{\lambda+1} \otimes v_+) \\
&= \mathsf{e}\mathbf{v}_{\lambda+1} \otimes v_- + \mathbf{v}_{\lambda+1} \otimes \mathsf{e}v_- + a(\lambda)(\mathsf{ef}\mathbf{v}_{\lambda+1} \otimes v_+ + \mathsf{f}\mathbf{v}_{\lambda+1} \otimes \mathsf{e}v_+).
\end{aligned}$$

But $\mathsf{e}\mathbf{v}_{\lambda+1} = \mathsf{e}v_+ = 0, \mathsf{e}v_- = v_+$, while $\mathsf{ef} = \mathsf{h} + \mathsf{fe}$ and $\mathsf{h}\mathbf{v}_{\lambda+1} = (\lambda+1)\mathbf{v}_{\lambda+1}$, so we obtain

$$0 = \mathbf{v}_{\lambda+1} \otimes v_+ + a(\lambda)(\lambda+1)\mathbf{v}_{\lambda+1} \otimes v_+.$$

Hence,

$$\begin{aligned} 0 &= 1 + a(\lambda)(\lambda+1) \\ \Rightarrow a(\lambda) &= \frac{-1}{\lambda+1}. \end{aligned}$$

Therefore, we conclude that

$$J_{VV}(v_+ \otimes v_-) = v_+ \otimes v_- + \frac{-1}{\lambda+1} v_- \otimes v_+.$$

So we see that we can write

$$[J_{VV}(\lambda)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{\lambda+1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the ordered basis of $V \otimes V$ is

$$(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-).$$

Therefore,

$$\begin{aligned} [R_{VV}(\lambda)] &= [J_{VV}(\lambda)]^{-1}[J^{21}_{VV}(\lambda)] = [J_{VV}(\lambda)]^{-1}[J_{VV}(\lambda)]^{\mathrm{T}} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{\lambda+1} & 0 \\ 0 & \frac{1}{\lambda+1} & 1 - \frac{1}{(\lambda+1)^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

3.5 The ABRR equation

Let $\mathfrak{g}$ be a semisimple Lie algebra. To each $\lambda \in \mathfrak{h}^*$, we can associate $\bar{\lambda} \in \mathfrak{h}$ so that $\bar{\lambda}\big|_{V[\mu]} = \langle \lambda, \mu \rangle \,\mathsf{Id}$. We let $\{\mathsf{x}_i\}$ be an orthonormal basis for $\mathfrak{h}$, and for each $\alpha \in \mathsf{R}$, we pick e_α such that $\langle \mathsf{e}_\alpha, \mathsf{e}_{-\alpha} \rangle = 1$.

Theorem 3.18 (ABRR equation, Arnaudon *et al.* (1998)) *Let V, W be any finite-dimensional representations of $\mathfrak{g}$, and let $\theta(\lambda) = \bar{\lambda} + \bar{\rho} - \frac{1}{2}\sum_i \mathsf{x}_i^2 \in \mathfrak{U}(\mathfrak{h})$. Then,*

$$[J_{VW}(\lambda), \mathsf{Id} \otimes \theta(\lambda)] = \sum_{\alpha \in \mathsf{R}_+} (\mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha) J_{VW}(\lambda). \tag{3.13}$$

Moreover, $J_{VW}(\lambda)$ is the unique solution of (3.13) in $\mathsf{End}_{\mathfrak{h}}(V \otimes W)$ of the form $\mathsf{Id} + \sum_{\beta>0}\sum_j \phi_j^\beta \otimes \psi_j^\beta$, where $\phi_j^\beta \in (\mathsf{End}\, V)[-\beta], \psi_j^\beta \in (\mathsf{End}\, W)[\beta]$.

Proof We look for a solution of (3.13) of the form

$$\mathrm{Id} + N(\lambda) \quad \text{where } N(\lambda) = \sum_{\beta>0}\sum_{j=1}^{m_\beta} \phi_j^\beta \otimes \psi_j^\beta, \phi_j^\beta \in (\mathrm{End}\, V)[-\beta], \psi_j^\beta \in (\mathrm{End}\, W)[\beta].$$

We have $[N(\lambda), \mathrm{Id} \otimes \theta(\lambda)] = \left(\sum_{\alpha \in \mathrm{R}_+} \mathrm{e}_{-\alpha} \otimes \mathrm{e}_\alpha\right)(\mathrm{Id} + N(\lambda))$. Thus,

$$\mathrm{ad}(\mathrm{Id} \otimes \theta(\lambda))N(\lambda) = -\left(\sum_{\alpha \in \mathrm{R}_+} \mathrm{e}_{-\alpha} \otimes \mathrm{e}_\alpha\right)(\mathrm{Id} + N(\lambda)).$$

We claim that, for generic λ, the operator $\mathrm{ad}\,\theta(\lambda) : (\mathrm{End}\, W)[\beta] \to (\mathrm{End}\, W)[\beta]$ is invertible for all $\beta > 0$. To prove this, we note that

$$\theta(\lambda)|_{W[\gamma]} = \left(\langle \lambda + \rho, \gamma\rangle - \tfrac{1}{2}\langle \gamma, \gamma\rangle\right) \mathrm{Id},$$

and that for generic λ, we have

$$\langle \lambda + \rho, \gamma_1\rangle - \tfrac{1}{2}\langle \gamma_1, \gamma_1\rangle \neq \langle \lambda + \rho, \gamma_2\rangle - \tfrac{1}{2}\langle \gamma_2, \gamma_2\rangle \quad \text{whenever } \gamma_1 \neq \gamma_2.$$

This implies that the operator

$$\mathrm{ad}\,\theta(\lambda) : \mathrm{Hom}_{\mathbf{C}}(W[\gamma_1], W[\gamma_2]) \to \mathrm{Hom}_{\mathbf{C}}(W[\gamma_1], W[\gamma_2])$$

is given by

$$\mathrm{ad}\,\theta(\lambda)|_{\mathrm{Hom}_{\mathbf{C}}(W[\gamma_1], W[\gamma_2])} = \left(\langle \lambda + \rho, \gamma_2\rangle - \frac{\langle \gamma_2, \gamma_2\rangle}{2} - \langle \lambda + \rho, \gamma_1\rangle + \frac{\langle \gamma_1, \gamma_1\rangle}{2}\right) \mathrm{Id}$$

and hence is invertible whenever $\gamma_1 \neq \gamma_2$. But

$$(\mathrm{End}\, W)[\beta] = \oplus_\gamma \mathrm{Hom}(W[\gamma], W[\gamma + \beta]),$$

so $\mathrm{ad}\,\theta(\lambda)$ is invertible on $(\mathrm{End}\, W)[\beta]$ whenever $\beta \neq 0$. So we can define an endomorphism of the space $\mathrm{End}V[\beta]$ by the formula

$$A(X) = -(1 \otimes (\mathrm{ad}\,\theta(\lambda))^{-1}) \sum_{\alpha \in \mathrm{R}_+} (\mathrm{e}_{-\alpha} \otimes \mathrm{e}_\alpha)(1 + X).$$

It is easy to see that $N(\lambda)$ is a solution of (3.13) if and only if $N(\lambda) = A(N(\lambda))$. So we must prove that A has a unique fixed point of the form

$$N(\lambda) = \sum_{\beta>0}\sum_{j=1}^{m_\beta} \phi_j^\beta \otimes \psi_j^\beta, \quad \phi_j^\beta \in (\mathrm{End}\, V)[-\beta], \psi_j^\beta \in (\mathrm{End}\, W)[\beta].$$

To see this, we define the *height* of $\beta \in \mathrm{Q}_+$ (denoted $\mathrm{ht}(\beta)$) to be the largest number of positive roots that sum to β. We then note that for all $m \geq 0$, and

$$X = \sum_{\beta>0} \sum_{j=1}^{m_\beta} \phi_j^\beta \otimes \psi_j^\beta, \quad \phi_j^\beta \in (\mathsf{End}\, V)[-\beta], \psi_j^\beta \in (\mathsf{End}\, W)[\beta],$$

one has

$$A^m(X) = A^m(0) + \text{ terms in } (\mathsf{End}\, V)[-\beta] \otimes (\mathsf{End}\, W)[\beta], \text{ with } \mathsf{ht}(\beta) > m.$$

Since V, W are finite-dimensional, it follows that, as $m \to \infty$, $A^m(X)$ stablilizes to an element of $\oplus_{\beta\in\mathsf{Q}_+}(\mathsf{End}\, V)[-\beta] \otimes (\mathsf{End}\, W)[\beta]$, and that this element is independent of X. It follows that A has a unique fixed point in $\oplus_{\beta\in\mathsf{Q}_+}(\mathsf{End}\, V)[-\beta] \otimes (\mathsf{End}\, W)[\beta]$, and this completes the proof. □

Proof that the fusion operator satisfies (3.13) Recall the Casimir operator,

$$\mathsf{C} = \sum_{\alpha\in\mathsf{R}_+} (\mathsf{e}_\alpha \mathsf{e}_{-\alpha} + \mathsf{e}_{-\alpha}\mathsf{e}_\alpha) + \sum_i \mathsf{x}_i^2,$$

where $\{\mathsf{x}_i\}$ is an orthonormal basis of $\mathfrak{h}$. Since $[\mathsf{e}_\alpha, \mathsf{e}_{-\alpha}] = \bar{\alpha}$, we see that

$$\mathsf{C} = 2 \sum_{\alpha\in\mathsf{R}_+} \mathsf{e}_{-\alpha}\mathsf{e}_\alpha + 2\bar{\rho} + \sum_i \mathsf{x}_i^2. \tag{3.14}$$

Now let V, W be finite-dimensional representations of $\mathfrak{g}$, and let $v \in V, w \in W$ be homogeneous elements. Define

$$F(\lambda) = \left\langle v^*_{\lambda-\mathsf{wt}\, w-\mathsf{wt}\, v} \otimes \mathsf{Id}, (\Phi^v_{\lambda-\mathsf{wt}\, w} \otimes \mathsf{Id})(\mathsf{C}\otimes\mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda \right\rangle.$$

We will compute $F(\lambda)$ in two different ways.

On the one hand, we know that C acts as a scalar on $M_{\lambda-\mathsf{wt}\, w}$; thus,

$$F(\lambda) = \langle \lambda - \mathsf{wt}\, w, \lambda - \mathsf{wt}\, w + 2\rho\rangle\, J_{VW}(\lambda)(v\otimes w). \tag{3.15}$$

On the other hand, we can compute the same quantity using expression (3.14) for C:

$$\begin{aligned} F(\lambda) = \Bigg\langle & v^*_{\lambda-\mathsf{wt}\, w-\mathsf{wt}\, v} \otimes \mathsf{Id}, (\Phi^v_{\lambda-\mathsf{wt}\, w} \otimes \mathsf{Id}) \\ & \left(\left(\sum_{\alpha\in\mathsf{R}_+} 2\mathsf{e}_{-\alpha}\mathsf{e}_\alpha + 2\bar{\rho} + \sum_i \mathsf{x}_i^2\right) \otimes \mathsf{Id}\right) \Phi^w_\lambda \mathbf{v}_\lambda \Bigg\rangle \\ = & \left\langle v^*_{\lambda-\mathsf{wt}\, w-\mathsf{wt}\, v} \otimes \mathsf{Id}, (A+B+C)(\Phi^v_{\lambda-\mathsf{wt}\, w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda \right\rangle, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} A &= 2(\mathsf{e}_{-\alpha})_1(\mathsf{e}_\alpha)_1 + 2(\mathsf{e}_{-\alpha})_1(\mathsf{e}_\alpha)_2 + 2(\mathsf{e}_{-\alpha})_2(\mathsf{e}_\alpha)_1 + 2(\mathsf{e}_{-\alpha})_2(\mathsf{e}_\alpha)_2, \\ B &= 2\bar{\rho}_1 + 2\bar{\rho}_2, \\ C &= \sum_i (\mathsf{x}_i)_1(\mathsf{x}_i)_1 + (\mathsf{x}_i)_1(\mathsf{x}_i)_2 + (\mathsf{x}_i)_2(\mathsf{x}_i)_1 + (\mathsf{x}_i)_2(\mathsf{x}_i)_2, \end{aligned}$$

and subscripts 1, 2 stands for the component in which the corresponding element acts. We then note that for all $u \in M_{\lambda-\mathsf{wt}\,w-\mathsf{wt}\,v}, w' \in W, v' \in V, \alpha \in \mathsf{R}$, we have

$$\left\langle v^*_{\lambda-\mathsf{wt}\,w-\mathsf{wt}\,v} \otimes \mathsf{Id}, (\mathsf{e}_{-\alpha})_1 u \otimes v' \otimes w' \right\rangle = 0. \tag{3.17}$$

Also, for all $\alpha \in \mathsf{R}$, we have

$$0 = (\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathsf{e}_\alpha \mathbf{v}_\lambda = ((\mathsf{e}_\alpha)_1 + (\mathsf{e}_\alpha)_2 + (\mathsf{e}_\alpha)_3)\,(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda,$$

and hence

$$((\mathsf{e}_\alpha)_1 + (\mathsf{e}_\alpha)_2)\,(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda = -(\mathsf{e}_\alpha)_3(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda. \tag{3.18}$$

We also note that

$$\begin{aligned}\langle \lambda, \rho\rangle\,(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda &= (\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \bar{\rho}\mathbf{v}_\lambda \\ &= (\bar{\rho}_1 + \bar{\rho}_2 + \bar{\rho}_3)(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda\end{aligned}$$

$$\implies (\bar{\rho}_1 + \bar{\rho}_2)(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda = -\bar{\rho}_3(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda. \tag{3.19}$$

We also have

$$\begin{aligned}(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \langle \lambda, \lambda\rangle\,\mathbf{v}_\lambda &= (\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \sum_i \mathsf{x}_i^2 \mathbf{v}_\lambda \\ &= \sum_i ((\mathsf{x}_i)_1 + (\mathsf{x}_i)_2 + (\mathsf{x}_i)_3)^2\,(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_i ((\mathsf{x}_i)_1 + (\mathsf{x}_i)_2)^2\,(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda \\ &\quad= \sum_i (((\mathsf{x}_i)_1 + (\mathsf{x}_i)_2 + (\mathsf{x}_i)_3)^2 + (\mathsf{x}_i)_3^2 - 2(\mathsf{x}_i)_3\,((\mathsf{x}_i)_1 + (\mathsf{x}_i)_2)(\mathsf{x}_i)_3)) \\ &\qquad \times (\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda \\ &\quad= (\langle \lambda, \lambda\rangle + \sum_i (\mathsf{x}_i)_3^2)(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda - 2\sum_i (\mathsf{x}_i)_3(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathsf{x}_i \mathbf{v}_\lambda \\ &\quad= \left(\langle \lambda, \lambda\rangle + \sum_i (\mathsf{x}_i)_3^2\right)(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda \qquad (3.20) \\ &\qquad - 2\sum_i (\mathsf{x}_i)_3 \lambda(\mathsf{x}_i)(\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda.\end{aligned}$$

Hence,

$$\left\langle v^*_{\lambda-\mathsf{wt}\,w-\mathsf{wt}\,v}\otimes \mathsf{Id}, \sum_i ((\mathsf{x}_i)_1+(\mathsf{x}_i)_2)^2\,(\Phi^v_{\lambda-\mathsf{wt}\,w}\otimes\mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda\right\rangle$$
$$= \left(\langle\lambda,\lambda\rangle + (\mathsf{Id}\otimes\sum_i \mathsf{x}_i^2) - 2(\mathsf{Id}\otimes\bar{\lambda})\right) J_{VW}(\lambda)(v\otimes w). \qquad (3.21)$$

Using equations (3.17), (3.18), (3.19) and (3.21), (3.16) becomes

$$F(\lambda) = \Bigg(-2\left(\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha\right) + \langle\lambda,\lambda+2\rho\rangle$$
$$-2(\mathsf{Id}\otimes(\bar{\lambda}+\bar{\rho})) + \left(\mathsf{Id}\otimes\sum_i \mathsf{x}_i^2\right)\Bigg) J_{VW}(\lambda)(v\otimes w). \qquad (3.22)$$

Equating (3.15) and (3.22) gives

$$\Bigg(\frac{1}{2}\langle\mathsf{wt}\,w,\mathsf{wt}\,w\rangle - \langle\mathsf{wt}\,w,\lambda+\rho\rangle$$
$$+\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha + (\mathsf{Id}\otimes\theta(\lambda))\Bigg) J_{VW}(\lambda)(v\otimes w) = 0,$$

and hence

$$\left(\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha\right) J_{VW}(\lambda)(v\otimes w)$$
$$= (\mathsf{Id}\otimes\theta(\lambda))J_{VW}(\lambda)(v\otimes w) - J_{VW}(\lambda)(\mathsf{Id}\otimes\theta(\lambda))(v\otimes w).$$

This completes the proof. □

3.6 The universal fusion and exchange operators

Proposition 3.19 *There exists a unique solution $J(\lambda)$ of the ABRR equation of the form*

$$\mathsf{Id} + (\sum(\mathsf{wt}<0)\otimes(\mathsf{wt}>0))$$

in a completion of $(\mathfrak{U}(\mathfrak{n}_+)\otimes\mathfrak{U}(\mathfrak{b}_-))^{\mathfrak{h}}$.

Proof Similar to the proof of Theorem 3.18. □

Definition 3.20 *This universal solution of the ABRR equation is called the universal fusion operator, denoted $J(\lambda)$.*

Definition 3.21 *The universal exchange operator is $R(\lambda) = J(\lambda)^{-1}J^{21}(\lambda)$.*

Example 3.22 Let us calculate the universal fusion operator $J(\lambda)$ for $\mathfrak{g} = \mathfrak{sl}_2 = \langle \mathsf{h}, \mathsf{e}, \mathsf{f} \rangle$.

We can write $J(\lambda) = \sum_{n=0}^{\infty} J_n$, where $J_n \in \mathfrak{U}(\mathfrak{g})[-2n] \otimes \mathfrak{U}(\mathfrak{g})[2n]$, $J_0 = \mathsf{Id}$. We note that for $\mathfrak{sl}_2$, we have $\bar{\lambda} = \frac{1}{2}\lambda \mathsf{h}$, $\rho = 1$, $\bar{\rho} = \frac{1}{2}\mathsf{h}$ and $x_1 = \frac{1}{\sqrt{2}}\mathsf{h}$. Thus,

$$\theta(\lambda) = \tfrac{1}{2}\left((\lambda+1)\,\mathsf{h} - \tfrac{1}{2}\mathsf{h}^2\right).$$

From this it follows that $[J(\lambda), \mathsf{Id} \otimes \frac{1}{2}\left((\lambda+1)\,\mathsf{h} - \frac{1}{2}\mathsf{h}^2\right)] = J(\lambda)$, and hence

$$\left[J_n, \mathsf{Id} \otimes \tfrac{1}{2}\left((\lambda+1)\,\mathsf{h} - \tfrac{1}{2}\mathsf{h}^2\right)\right] = (\mathsf{f} \otimes \mathsf{e})J_{n-1} \quad \text{for all } n \geq 1. \tag{3.23}$$

Now,

$$\begin{aligned}
\left[\mathsf{e}^n, (\lambda+1)\mathsf{h} - \tfrac{1}{2}\mathsf{h}^2\right] &= -2n(\lambda+1)\mathsf{e}^n + \tfrac{1}{2}[\mathsf{h}^2, \mathsf{e}^n] \\
&= -2n(\lambda+1)\mathsf{e}^n + \tfrac{1}{2}\left(\mathsf{h}[\mathsf{h}, \mathsf{e}^n] + [\mathsf{h}, \mathsf{e}^n]\mathsf{h}\right) \\
&= -2n(\lambda+1)\mathsf{e}^n + n\mathsf{h}\mathsf{e}^n + n\mathsf{e}^n\mathsf{h} \\
&= \left(-2n(\lambda+1) + 2n\mathsf{h} - 2n^2\right)\mathsf{e}^n.
\end{aligned}$$

From this and from (3.23) it follows that

$$\begin{aligned}
(\mathsf{f} \otimes \mathsf{e})J_{n-1} &= \left[J_n, \mathsf{Id} \otimes \tfrac{1}{2}\left((\lambda+1)\,\mathsf{h} - \tfrac{1}{2}\mathsf{h}^2\right)\right] \\
&= -n\left(\mathsf{Id} \otimes (\lambda + 1 - \mathsf{h} + n)\right) J_n.
\end{aligned}$$

Hence,

$$J_n = -\frac{1}{n}\left(\mathsf{f} \otimes \frac{1}{\lambda + 1 - \mathsf{h} + n}\mathsf{e}\right) J_{n-1},$$

and so

$$J_n = \frac{(-1)^n}{n!}\mathsf{f}^n \otimes \prod_{j=1}^{n} \frac{1}{\lambda + j - \mathsf{h} + n}\mathsf{e}^n.$$

Therefore,

$$J(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mathsf{f}^n \otimes \prod_{j=1}^{n} \frac{1}{\lambda + j - \mathsf{h} + n}\mathsf{e}^n.$$

Remark 3.23 If $V = W = \mathbf{C}^2$, then $\mathsf{f}^2 = \mathsf{e}^2 = 0$, so

$$\begin{aligned}
J(v_+ \otimes v_-) &= \left(1 - \mathsf{f} \otimes \frac{1}{\lambda - \mathsf{h} + 2}\mathsf{e}\right)(v_+ \otimes v_-) \\
&= v_+ \otimes v_- - \frac{1}{\lambda + 1}(v_- \otimes v_+).
\end{aligned}$$

This agrees with the result obtained in Example 3.17.

4

QUANTUM GROUPS

The material in this chapter is standard. It can be found in textbooks on quantum groups, such as Kassel (1995), Etingof and Schiffmann (2002), Jantzen (1996), Lusztig (1993).

4.1 Hopf algebras

Definition 4.1 *A Hopf algebra* $\mathbf{H}$ *over a field* $\mathbf{k}$ *is a vector space over* $\mathbf{k}$ *along with five operations:*

- $\mathbf{m} : \mathbf{H} \otimes \mathbf{H} \to \mathbf{H}$ *(multiplication),*
- $\mathbf{\Delta} : \mathbf{H} \to \mathbf{H} \otimes \mathbf{H}$ *(comultiplication),*
- $i : \mathbf{k} \to \mathbf{H}$ *(unit),*
- $\epsilon : \mathbf{H} \to \mathbf{k}$ *(counit),*
- $\mathbf{S} : \mathbf{H} \to \mathbf{H}$ *(antipode);*

satisfying the following seven axioms:

1. $(\mathbf{m} \otimes \mathsf{Id})\mathbf{m} = (\mathsf{Id} \otimes \mathbf{m})\mathbf{m}$,
2. $\mathbf{m}(\mathsf{Id} \otimes i) = \mathbf{m}(i \otimes \mathsf{Id}) = \mathsf{Id}$,
3. $(\mathbf{\Delta} \otimes \mathsf{Id})\mathbf{\Delta} = (\mathsf{Id} \otimes \mathbf{\Delta})\mathbf{\Delta}$,
4. $(\epsilon \otimes \mathsf{Id})\mathbf{\Delta} = (\mathsf{Id} \otimes \epsilon)\mathbf{\Delta} = \mathsf{Id}$,
5. $\mathbf{m}(\mathsf{Id} \otimes \mathbf{S})\mathbf{\Delta} = \mathbf{m}(\mathbf{S} \otimes \mathsf{Id})\mathbf{\Delta} = i \circ \epsilon$,
6. $\mathbf{\Delta} : \mathbf{H} \to \mathbf{H} \otimes \mathbf{H}$ *is an algebra homomorphism,*
7. $\epsilon : \mathbf{H} \to \mathbf{k}$ *is an algebra homomorphism.*

Example 4.2 Let G be a finite group, and let $\mathbf{H} = F(G)$ be the set of maps from G to $\mathbf{C}$. We know that $\mathbf{H} \otimes \mathbf{H} = F(G) \otimes F(G) = F(G \times G)$. Let $\mathbf{m}$ be multiplication of functions, $i(1) = 1, \mathbf{\Delta}(f)(g, h) = f(gh), \epsilon(f) = f(1_G)$ and $(\mathbf{S}f)(g) = f(g^{-1})$. Then $\mathbf{H}$ is a Hopf algebra.

Theorem 4.3 *If* $\mathbf{H}$ *is a finite-dimensional commutative Hopf algebra over an algebraically closed field of characteristic 0, then* $\mathbf{H} = F(G)$ *for a unique finite group* G*.*

In light of this theorem, we will need to drop the commutativity assumption in order to obtain more examples.

Proposition 4.4 *Let* $(\mathbf{H}, \mathbf{m}, i, \boldsymbol{\Delta}, \epsilon, \mathbf{S})$ *be a Hopf algebra. Then*

$$(\mathbf{H}^*, \mathbf{m}_*, i_*, \boldsymbol{\Delta}_*, \epsilon_*, \mathbf{S}_*),$$

where $\mathbf{m}_* = \boldsymbol{\Delta}^*, \boldsymbol{\Delta}_* = \mathbf{m}^*, \mathbf{S}_* = \mathbf{S}^*, i_* = \epsilon^*, \epsilon_* = i^*$, *is also a Hopf algebra.*

Example 4.5 If $\mathbf{H} = F(G)$ for some finite group G, then $\mathbf{H}^*$ is the group algebra $\mathbf{k}[G]$, with $\mathbf{m}_* =$ multiplication in $\mathbf{k}[G]$, $i_*(1) = 1_G, \boldsymbol{\Delta}_*(g) = g \otimes g, \epsilon_*(g) = 1, \mathbf{S}_*(g) = g^{-1}$ for all $g \in G$.

Because of this, we make the following definition.

Definition 4.6 *In any Hopf algebra* $\mathbf{H}$*, a nonzero element* $g \in \mathbf{H}$ *such that* $\boldsymbol{\Delta}(g) = g \otimes g$ *is called a grouplike element.*

Example 4.7 Let $\mathfrak{g}$ be a Lie algebra, and let $\mathbf{H} = \mathfrak{U}(\mathfrak{g})$. Let $\mathbf{m}$ be the usual multiplication in $\mathfrak{U}(\mathfrak{g})$, $i(1) = 1_{\mathfrak{U}(\mathfrak{g})}, \boldsymbol{\Delta}(\mathsf{x}) = \mathsf{x} \otimes 1 + 1 \otimes \mathsf{x}, \epsilon(\mathsf{x}) = 0$ and $\mathbf{S}(\mathsf{x}) = -\mathsf{x}$ for all $\mathsf{x} \in \mathfrak{g}$. Then $\mathbf{H}$ is a Hopf algebra.

Notation 4.8 $\boldsymbol{\Delta}^{\mathrm{op}}$ *is the algebra homomorphism defined as follows: if* $\boldsymbol{\Delta}(a) = \sum_i b_i \otimes c_i$*, then* $\boldsymbol{\Delta}^{\mathrm{op}}(a) = \sum_i c_i \otimes b_i$.

Proposition 4.9 *In any Hopf algebra* $\mathbf{H}$*,* $\mathbf{S} : \mathbf{H} \to \mathbf{H}$ *is an algebra and coalgebra antihomomorphism; that is,*

$$\mathbf{S}(ab) = \mathbf{S}(b)\mathbf{S}(a) \text{ and } \boldsymbol{\Delta}\mathbf{S}(a) = (\mathbf{S} \otimes \mathbf{S})\boldsymbol{\Delta}^{\mathrm{op}}(a) \text{ for all } a, b \in \mathbf{H}.$$

Remark 4.10 In the examples that we have seen so far, we have $\mathbf{S}^2 = \mathsf{Id}$; however, as we will see, this is not always true.

4.2 Representations of Hopf algebras

Given a Hopf algebra $\mathbf{H}$, consider the category $\mathsf{Rep}\,\mathbf{H}$ of left $\mathbf{H}$-modules (here we regard $\mathbf{H}$ as an algebra, taking only $\mathbf{m}$ and i into consideration).

We can use the additional structure of $\mathbf{H}$ to define the following.

Definition 4.11 *Given vector spaces* V, W *and representations* π_V, π_W*, we can use* $\boldsymbol{\Delta}$ *to define the tensor product of representations on* $V \otimes W$*:* $\pi_{V\otimes W} : \mathbf{H} \to \mathsf{End}(V \otimes W) \cong \mathsf{End}\,V \otimes \mathsf{End}\,W$ *is defined by* $\pi_{V\otimes W} = (\pi_V \otimes \pi_W)\boldsymbol{\Delta}$.

Obviously, this generalizes the definition of tensor product for groups and Lie algebras.

Proposition 4.12 *$(V\otimes W)\otimes U = V\otimes(W\otimes U)$ for all representations V, W, U of* $\mathbf{H}$.

Proof This is a simple consequence of axiom 3. □

Definition 4.13 *Using ϵ, we can define the trivial representation:*

$$V_{\mathrm{tr}} = \mathbf{C}, \pi_{V_{\mathrm{tr}}}(x) = \epsilon(x).$$

Proposition 4.14 *$V_{\mathrm{tr}} \otimes W = W = W \otimes V_{\mathrm{tr}}$ for all representations W of* $\mathbf{H}$.

Proof This is a simple consequence of axiom 4. □

Definition 4.15 *Using $\mathbf{S}$, we can define the dual representation: given a representation π_V on a vector space V, we define $\pi_{V^*} : V^* \to V^*$ by $\pi_{V^*}(x) = (\pi_V(\mathbf{S}(x)))^*$.*

Remark 4.16 If $\mathbf{S}$ is invertible, we can define the *left dual*,

$$\pi_{{}^*V}(x) = (\pi_V(\mathbf{S}^{-1}(x)))^*.$$

Because of Remark 4.10, this is *not* the same as the dual of Definition 4.15. We will then have ${}^*V^* = V$. However, in general, $V^{**} \neq V$.

4.3 The quantum group $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$

Let us consider $\mathfrak{sl}_2$. We have $\mathsf{h}, \mathsf{e}, \mathsf{f} \in \mathfrak{sl}_2$ such that $[\mathsf{h}, \mathsf{e}] = 2\mathsf{e}, [\mathsf{h}, \mathsf{f}] = -2\mathsf{f}, [\mathsf{e}, \mathsf{f}] = \mathsf{h}$. $\mathbf{C}\mathsf{h} = \mathfrak{h} \subset \mathfrak{sl}_2$ is the Cartan subalgebra. Let $\alpha \in \mathfrak{h}^*$ be the unique positive root. Let $\mathsf{q} \in \mathbf{C}, \mathsf{q} \neq \pm 1$.

Definition 4.17 *The quantum group $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$ is generated by $\mathsf{E}, \mathsf{F}, \mathsf{q}^{\mathsf{x}} (\mathsf{x} \in \mathfrak{h})$. The relations are*

$$\mathsf{q}^{\mathsf{x}+\mathsf{y}} = \mathsf{q}^{\mathsf{x}}\mathsf{q}^{\mathsf{y}}, \quad \mathsf{q}^0 = 1, \quad \mathsf{q}^{\mathsf{x}}\mathsf{E}\mathsf{q}^{-\mathsf{x}} = \mathsf{q}^{\alpha(\mathsf{x})}\mathsf{E}, \quad \mathsf{q}^{\mathsf{x}}\mathsf{F}\mathsf{q}^{-\mathsf{x}} = \mathsf{q}^{-\alpha(\mathsf{x})}\mathsf{F}$$

$$\text{and} \quad [\mathsf{E}, \mathsf{F}] = \frac{\mathsf{q}^{\mathsf{h}} - \mathsf{q}^{-\mathsf{h}}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

We can formally calculate the following limit:

$$\lim_{\mathsf{q}\to 1} \frac{\mathsf{q}^{\mathsf{h}} - \mathsf{q}^{-\mathsf{h}}}{\mathsf{q} - \mathsf{q}^{-1}} = \mathsf{h};$$

thus, $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$ should be thought of as a deformation of $\mathfrak{U}(\mathfrak{sl}_2)$. Later, we will make this idea more precise.

Theorem 4.18 *There exists a unique Hopf algebra structure on $\mathfrak{U}_\mathsf{q}(\mathfrak{sl}_2)$, given by*

- $\boldsymbol{\Delta}(\mathsf{q}^\mathsf{x}) = \mathsf{q}^\mathsf{x} \otimes \mathsf{q}^\mathsf{x}$ *(thus q^x is a grouplike element);*
- $\boldsymbol{\Delta}(\mathsf{E}) = \mathsf{E} \otimes \mathsf{q}^\mathsf{h} + 1 \otimes \mathsf{E}$*;*
- $\boldsymbol{\Delta}(\mathsf{F}) = \mathsf{F} \otimes 1 + \mathsf{q}^{-\mathsf{h}} \otimes \mathsf{F}$*;*
- $\epsilon(\mathsf{q}^\mathsf{x}) = 1, \epsilon(\mathsf{E}) = \epsilon(\mathsf{F}) = 0$*;*

and there is only one way to define the antipode map.

We will calculate the antipode, knowing that $\mathbf{m}(\mathbf{S} \otimes 1)\boldsymbol{\Delta}(a) = i \circ \epsilon(a) = \mathbf{m}(1 \otimes \mathbf{S})\boldsymbol{\Delta}(a)$ for all a in a Hopf algebra.

Proposition 4.19 *Let $\mathbf{H}$ be any Hopf algebra, and let $g \in \mathbf{H}$ be such that $\boldsymbol{\Delta}(g) = g \otimes g$ and $\epsilon(g) = 1$. Then $\mathbf{S}(g) = g^{-1}$.*

Proof $1 = i \circ \epsilon(g) = \mathbf{m}(\mathbf{S} \otimes 1)\boldsymbol{\Delta}(g) = \mathbf{m}(\mathbf{S} \otimes 1)(g \otimes g) = \mathbf{S}(g)g$. Similarly, $1 = g\mathbf{S}(g)$. Therefore, $\mathbf{S}(g) = g^{-1}$. □

Computation of the antipode map In $\mathfrak{U}_\mathsf{q}(\mathfrak{sl}_2)$, we have (by proposition 4.19) $\mathbf{S}(\mathsf{q}^\mathsf{x}) = (\mathsf{q}^\mathsf{x})^{-1} = \mathsf{q}^{-\mathsf{x}}$. We also have $0 = i \circ \epsilon(\mathsf{E}) = \mathbf{m}(\mathbf{S} \otimes 1)\boldsymbol{\Delta}(\mathsf{E}) = \mathbf{m}(\mathbf{S}(\mathsf{E}) \otimes \mathsf{q}^\mathsf{h} + 1 \otimes \mathsf{E}) = \mathbf{S}(\mathsf{E})\mathsf{q}^\mathsf{h} + \mathsf{E}$; thus, $\mathbf{S}(\mathsf{E}) = -\mathsf{E}\mathsf{q}^{-\mathsf{h}}$. Similarly, $\mathbf{S}(\mathsf{F}) = -\mathsf{q}^{-\mathsf{h}}\mathsf{F}$. □

Remark 4.20 We also have $\mathbf{S}^2(\mathsf{q}^\mathsf{x}) = \mathbf{S}(\mathsf{q}^{-\mathsf{x}}) = \mathsf{q}^\mathsf{x}, \mathbf{S}^2(\mathsf{E}) = \mathbf{S}(-\mathsf{E}\mathsf{q}^{-\mathsf{h}}) = -\mathbf{S}(\mathsf{q}^{-\mathsf{h}})\mathbf{S}(\mathsf{E}) = \mathsf{q}^\mathsf{h}\mathsf{E}\mathsf{q}^{-\mathsf{h}}$, and, similarly, $\mathbf{S}^2(\mathsf{F}) = \mathsf{q}^\mathsf{h}\mathsf{F}\mathsf{q}^{-\mathsf{h}}$. Thus, $\mathbf{S}^2 = \mathsf{ad}(\mathsf{q}^\mathsf{h})$ (conjugation by q^h). In particular, $\mathbf{S}^2 \neq \mathsf{Id}$.

4.4 The quantum group $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$

Let $\mathfrak{g}$ be an arbitrary simple Lie algebra, and let $A = (a_{ij})$ be its Cartan matrix. Recall that there exist unique relatively prime positive integers $d_i, i = 1, \ldots, r$ such that $d_i a_{ij} = d_j a_{ji}$. (In fact, we have $d_i = 2/\langle \alpha_i, \alpha_i \rangle$, where $\langle \cdot, \cdot \rangle$ is normalized so that $\langle \theta, \theta \rangle = 2$.) Let $\mathsf{q} \in \mathbf{C}, \mathsf{q} \neq \pm 1$. Choose a value of $\log(\mathsf{q})$, and for any number or operator A, set $\mathsf{q}^A \overset{\text{def}}{=} e^{A \log(\mathsf{q})}$.

Definition 4.21 *Let $x \in \mathbf{C}$.*

- *The q-analog of x is*
$$[x]_\mathsf{q} = \frac{\mathsf{q}^x - \mathsf{q}^{-x}}{\mathsf{q} - \mathsf{q}^{-1}}.$$
- *The q-analog of the factorial is*
$$[n]_\mathsf{q}! = \prod_{l=1}^{n} [l]_\mathsf{q} = \frac{(\mathsf{q} - \mathsf{q}^{-1}) \cdots (\mathsf{q}^n - \mathsf{q}^{-n})}{(\mathsf{q} - \mathsf{q}^{-1})^n}.$$

Definition 4.22 *The quantum group $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ is generated by $\mathsf{E}_i, \mathsf{F}_i, \mathsf{q}^{\mathsf{x}}(\mathsf{x} \in \mathfrak{h})$. The relations are*

$$\mathsf{q}^{\mathsf{x}+\mathsf{y}} = \mathsf{q}^{\mathsf{x}}\mathsf{q}^{\mathsf{y}}, \mathsf{q}^0 = 1, \mathsf{q}^{\mathsf{x}}\mathsf{E}_i\mathsf{q}^{-\mathsf{x}} = \mathsf{q}^{\alpha_i(\mathsf{x})}\mathsf{E}_i, \mathsf{q}^{\mathsf{x}}\mathsf{F}_i\mathsf{q}^{-\mathsf{x}} = \mathsf{q}^{-\alpha_i(\mathsf{x})}\mathsf{F}_i,$$

$$[\mathsf{E}_i, \mathsf{F}_j] = \delta_{ij}\frac{\mathsf{q}^{d_i\mathsf{h}_i} - \mathsf{q}^{-d_i\mathsf{h}_i}}{\mathsf{q}^{d_i} - \mathsf{q}^{-d_i}}, \quad \textit{and the } \mathsf{q}\textit{-Serre relations:}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{\mathsf{q}}![1 - a_{ij} - k]_{\mathsf{q}}!}\mathsf{E}_i^{1-a_{ij}-k}\mathsf{E}_j\mathsf{E}_i^k = 0, \tag{4.1}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{\mathsf{q}}![1 - a_{ij} - k]_{\mathsf{q}}!}\mathsf{F}_i^{1-a_{ij}-k}\mathsf{F}_j\mathsf{F}_i^k = 0. \tag{4.2}$$

One may ask why we say that (4.1) and (4.2) are deformations of the Serre relations. To answer this question, denote by L_a, R_a the operators of left, respectively right multiplication by $a \in \mathfrak{g}$ in $\mathfrak{U}(\mathfrak{g})$; then the usual Serre relations can be written as

$$\begin{aligned} 0 &= (\mathsf{ad}\,\mathsf{e}_i)^{1-a_{ij}}\mathsf{e}_j \\ &= (L_{\mathsf{e}_i} - R_{\mathsf{e}_i})^{1-a_{ij}}\mathsf{e}_j \\ &= \sum_{k=0}^{1-a_{ij}} \frac{(-1)^k(1-a_{ij})!}{k!(1-a_{ij}-k)!}L_{\mathsf{e}_i}^{1-a_{ij}-k}R_{\mathsf{e}_i}^k\mathsf{e}_j \\ &= \sum_{k=0}^{1-a_{ij}} \frac{(-1)^k(1-a_{ij})!}{k!(1-a_{ij}-k)!}\mathsf{e}_i^{1-a_{ij}-k}\mathsf{e}_j\mathsf{e}_i^k. \end{aligned} \tag{4.3}$$

Clearly, (4.1) is obtained from (4.3) by replacing each factorial by its q^{d_i}-analog. This explains why (4.1) is a deformation of (4.3), and a similar argument can be applied to (4.2).

It may seem like we could have defined the q-Serre relations in another way, but it is not the case. We cannot define the ideal of q-Serre relations in any other way if we want the PBW theorem to be satisfied.

4.5 PBW for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$

In the case of a Lie algebra, we have a decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

and the PBW theorem says, in particular, that the multiplication map

$$\mathfrak{U}(\mathfrak{n}_+) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}_-) \to \mathfrak{U}(\mathfrak{g}), \quad a \otimes b \otimes c \mapsto abc$$

is an isomorphism (in any of the six possible orders). For quantum groups, we define

$$\mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_+) = \langle E_i, i = 1, \ldots, r \rangle, \quad \mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_-) = \langle F_i, i = 1, \ldots, r \rangle, \quad \mathfrak{U}(\mathfrak{h}) = \langle \mathsf{q}^{\mathsf{x}}, \mathsf{x} \in \mathfrak{h} \rangle .$$

Then we have the following.

Proposition 4.23

1. *The multiplication map*

$$\mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_+) \otimes \mathfrak{U}_{\mathsf{q}}(\mathfrak{h}) \otimes \mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_-) \to \mathfrak{U}_{\mathsf{q}}(\mathfrak{g}), \quad a \otimes b \otimes c \mapsto abc$$

   is an isomorphism (in any of the six possible orders);
2. $\mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_+)$ *(respectively* $\mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_-)$*) is the free algebra generated by the* E_i *(respectively* F_i*), modulo the* q*-Serre relations for* $\mathsf{E}_i(\mathsf{F}_i)$*;*
3. *Moreover,* $\dim \mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_\pm)[\beta] = \dim \mathfrak{U}(\mathfrak{n}_\pm)[\beta]$ *for all* $\beta \in \mathsf{Q}_\pm$*. Thus* $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ *is a "flat deformation" of* $\mathfrak{U}(\mathfrak{g})$*.*

4.6 The Hopf algebra structure on $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$

Theorem 4.24 *There exists a unique Hopf algebra structure on* $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$*, given by*

- $\mathbf{\Delta}(\mathsf{q}^{\mathsf{h}}) = \mathsf{q}^{\mathsf{h}} \otimes \mathsf{q}^{\mathsf{h}}$;
- $\mathbf{\Delta}(\mathsf{E}_i) = \mathsf{E}_i \otimes \mathsf{q}^{d_i \mathsf{h}_i} + 1 \otimes \mathsf{E}_i$;
- $\mathbf{\Delta}(\mathsf{F}_i) = \mathsf{F}_i \otimes 1 + \mathsf{q}^{-d_i \mathsf{h}_i} \otimes \mathsf{F}_i$;
- $\epsilon(\mathsf{q}^{\mathsf{h}}) = 1, \epsilon(\mathsf{E}_i) = \epsilon(\mathsf{F}_i) = 0$;

for all $\mathsf{h} \in \mathfrak{h}, i = 1 \ldots r$*, and there is only one way to define the antipode map:* $\mathbf{S}(\mathsf{E}_i) = -\mathsf{E}_i \mathsf{q}^{-d_i \mathsf{h}_i}, \mathbf{S}(\mathsf{F}_i) = -\mathsf{q}^{-d_i \mathsf{h}_i} \mathsf{F}_i, \mathbf{S}(\mathsf{q}^{\mathsf{h}}) = \mathsf{q}^{-\mathsf{h}}$.

Remark 4.25 For $i = 1, \ldots, r$, there is a Hopf subalgebra

$$\mathfrak{U}_i = \left\langle \mathsf{E}_i, \mathsf{F}_i, \mathsf{q}^{t \mathsf{h}_i}, t \in \mathbf{C} \right\rangle,$$

which is isomorphic to $\mathfrak{U}_{\mathsf{q}^{d_i}}(\mathfrak{sl}_2)$.

Proposition 4.26 $\mathbf{S}^2 = \mathsf{Ad}\, \mathsf{q}^{2\bar{\rho}}$.

Proof For $\mathsf{h} \in \mathfrak{h}$, we have $\mathbf{S}^2(\mathsf{q}^{\mathsf{h}}) = \mathbf{S}(\mathsf{q}^{-\mathsf{h}}) = \mathsf{q}^{\mathsf{h}}$. For $i = 1, \ldots, r$, we have $\mathbf{S}^2(\mathsf{E}_i) = \mathbf{S}(-\mathsf{E}_i \mathsf{q}^{-d_i \mathsf{h}_i}) = -\mathbf{S}(\mathsf{q}^{-d_i \mathsf{h}_i})\mathbf{S}(\mathsf{E}_i) = \mathsf{q}^{d_i \mathsf{h}_i} \mathsf{E}_i \mathsf{q}^{-d_i \mathsf{h}_i} = \mathsf{q}^{2\bar{\rho}} \mathsf{E}_i \mathsf{q}^{-2\bar{\rho}}$. Similarly, $\mathbf{S}^2(\mathsf{F}_i) = \mathsf{q}^{2\bar{\rho}} \mathsf{F}_i \mathsf{q}^{-2\bar{\rho}}$. □

In particular, if V is a representation of $\mathfrak{U}_q(\mathfrak{g})$, then $\mathsf{q}^{2\bar{\rho}} : V \to V^{**}$ is an isomorphism of representations.

4.7 Representation theory of $\mathfrak{U}_q(\mathfrak{g})$

From now on, we assume that q is not a root of unity.

Recall that in the Lie algebra theory one defines the positive and negative Borel subalgebras $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$, $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{h}$. Similarly, for quantum groups we define quantized Borel subalgebras $\mathfrak{U}_q(\mathfrak{b}_+) \stackrel{\mathrm{def}}{=} \mathfrak{U}_q(\mathfrak{n}_+)\mathfrak{U}_q(\mathfrak{h})$, $\mathfrak{U}_q(\mathfrak{b}_-) \stackrel{\mathrm{def}}{=} \mathfrak{U}_q(\mathfrak{n}_-)\mathfrak{U}_q(\mathfrak{h})$. Note that unlike $\mathfrak{U}_q(\mathfrak{n}_\pm)$, they are Hopf subalgebras of $\mathfrak{U}_q(\mathfrak{g})$.

Let $\lambda \in \mathfrak{h}^*$. There is a map $\lambda : \mathfrak{U}_q(\mathfrak{b}_+) \to \mathbf{C}$, given by $\lambda(q^h) = q^{\lambda(h)}$, $\lambda(E_i) = 0$.

Definition 4.27 *The Verma module M_λ with highest weight λ is*

$$M_\lambda = \mathsf{Ind}_{\mathfrak{U}_q(\mathfrak{b}_+)}^{\mathfrak{U}_q(\mathfrak{g})} \lambda.$$

By PBW, we also have $M_\lambda = \mathfrak{U}_q(\mathfrak{n}_-)\mathbf{v}_\lambda$, with $q^h\mathbf{v}_\lambda = q^{\lambda(h)}\mathbf{v}_\lambda$, $E_i\mathbf{v}_\lambda = 0$.

The Verma module for $\mathfrak{U}_q(\mathfrak{g})$ shares many of the properties of the Verma module for $\mathfrak{g}$.

Proposition 4.28

1. *For generic λ, M_λ is irreducible.*
2. *If $\lambda \in P_+$, then $V_\lambda = M_\lambda/J_\lambda$ is a finite-dimensional irreducible module.*

Theorem 4.29 (Lusztig (1988), Rosso (1988))

1. *Any finite-dimensional representation of $\mathfrak{U}_q(\mathfrak{g})$ is a direct sum of irreducible representations.*
2. *The irreducible representation V_λ of $\mathfrak{U}_q(\mathfrak{g})$ has the same character as the representation V_λ of $\mathfrak{g}$, for all $\lambda \in P_+$.*
3. *If V is a finite-dimensional representation of $\mathfrak{U}_q(\mathfrak{g})$ with $V = \oplus_{\lambda\in\mathfrak{h}^*} V[\lambda]$, where $V[\lambda] = \{v \in V, q^h v = q^{\lambda(h)} v$ for all $h \in \mathfrak{h}\}$, then V is a direct sum of $V_\mu, \mu \in P_+$.*

There is also a Shapovalov form on M_λ for $\mathfrak{U}_q(\mathfrak{g})$, defined similarly to the case of $\mathfrak{U}(\mathfrak{g})$. Its determinant is given by a formula similar to the $q = 1$ case (see de Concini and Kač (1990)). This formula implies that for generic λ, M_λ is irreducible.

Example 4.30 ($\mathfrak{U}_q(\mathfrak{sl}_2)$) We have $M_\lambda = \langle \mathbf{v}_\lambda, F\mathbf{v}_\lambda, F^2\mathbf{v}_\lambda, \ldots \rangle$. The action of $\mathfrak{U}_q(\mathfrak{sl}_2)$ is:

- $FF^k\mathbf{v}_\lambda = F^{k+1}\mathbf{v}_\lambda$;
- $q^{th}F^k\mathbf{v}_\lambda = q^{t(\lambda-2k)}F^k\mathbf{v}_\lambda$;
- $EF^k\mathbf{v}_\lambda = c_k F^{k-1}\mathbf{v}_\lambda$, and we need to compute c_k.

Clearly, $c_0 = 0$. For $k \geq 1$, we have:

$$\begin{aligned}
\mathsf{E}\mathsf{F}^k\mathbf{v}_\lambda &= \left(\mathsf{F}\mathsf{E}\mathsf{F}^{k-1} + \frac{\mathsf{q}^{\mathsf{h}} - \mathsf{q}^{-\mathsf{h}}}{\mathsf{q} - \mathsf{q}^{-1}}\mathsf{F}^{k-1}\right)\mathbf{v}_\lambda \\
&= \left(c_{k-1} + \frac{\mathsf{q}^{\lambda-2(k-1)} - \mathsf{q}^{-\lambda+2(k-1)}}{\mathsf{q} - \mathsf{q}^{-1}}\right)\mathsf{F}^{k-1}\mathbf{v}_\lambda \\
\Longrightarrow c_k &= c_{k-1} + \frac{\mathsf{q}^{\lambda-2(k-1)} - \mathsf{q}^{-\lambda+2(k-1)}}{\mathsf{q} - \mathsf{q}^{-1}} \\
&= \sum_{j=1}^{k-1} \frac{\mathsf{q}^{\lambda-2j} - \mathsf{q}^{-\lambda+2j}}{\mathsf{q} - \mathsf{q}^{-1}} \\
&= \frac{\mathsf{q}^{\lambda}\frac{1-\mathsf{q}^{2k}}{1-\mathsf{q}^{-2}} - \mathsf{q}^{-\lambda}\frac{\mathsf{q}^{2k}-1}{\mathsf{q}^{2}-1}}{\mathsf{q} - \mathsf{q}^{-1}} \\
&= \frac{(\mathsf{q}^{k} - \mathsf{q}^{-k})(\mathsf{q}^{\lambda-k+1} - \mathsf{q}^{-\lambda+k-1})}{(\mathsf{q} - \mathsf{q}^{-1})^2} \\
&= [k]_{\mathsf{q}}[\lambda - k + 1]_{\mathsf{q}}.
\end{aligned}$$

Thus $\mathsf{E}\mathsf{F}^k\mathbf{v}_\lambda = [k]_{\mathsf{q}}[\lambda - k + 1]_{\mathsf{q}}\mathsf{F}^{k-1}\mathbf{v}_\lambda$. We note that $[k]_{\mathsf{q}} = 0 \iff \mathsf{q}^k = \mathsf{q}^{-k} \iff \mathsf{q}^{2k} = 1$, and $[\lambda - k + 1]_{\mathsf{q}} = 0 \iff \lambda = k - 1 + (2\pi i n/\hbar)$ for some $n \in \mathbf{Z}$. Therefore,

- If q is a root of unity, then all Verma modules are reducible.
- If q is not a root of unity, then M_λ is irreducible if and only if $\lambda \neq k - 1 + (2\pi i n/\hbar), k \in Z_+, n \in \mathbf{Z}$. If $\lambda \neq k - 1 + (2\pi i n/\hbar)$, then there is a finite-dimensional subrepresentation $\langle \mathbf{v}_\lambda, \mathsf{F}\mathbf{v}_\lambda, \ldots, \mathsf{F}^k\mathbf{v}_\lambda \rangle$.

Note again the similarity between M_λ for $\mathfrak{sl}_2$ and M_λ for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$: in the former case, we have $\mathsf{e}\mathsf{f}^k\mathbf{v}_\lambda = k(\lambda - k + 1)\mathsf{f}^{k-1}\mathbf{v}_\lambda$, while in the latter case, we have $\mathsf{E}\mathsf{F}^k\mathbf{v}_\lambda = [k]_{\mathsf{q}}[\lambda - k + 1]_{\mathsf{q}}\mathsf{F}^{k-1}\mathbf{v}_\lambda$. For $\mathfrak{sl}_2$, M_λ is irreducible if and only if $\lambda \notin \mathbf{Z}_+$. For $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$, M_λ is irreducible if and only if $\lambda \neq k + (2\pi i n/\hbar)$ for $k \in \mathbf{Z}_+, n \in \mathbf{Z}$.

4.8 Formal version of quantum groups

We will now take a somewhat different approach to quantum groups. Namely, we will take $\hbar$ to be a *formal parameter*, and will be working over the ring of formal power series $\mathbf{C}[[\hbar]]$. The parameter q will be defined as a power series:

$$\mathsf{q} = e^{\hbar/2} = \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \in \mathbf{C}[[\hbar]].$$

Definition 4.31 $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ *is an algebra over* $\mathbf{C}[[\hbar]]$, *topologically generated by*

$$E_i, \quad F_i (i = 1, \ldots, r), \quad \mathsf{h} \in \mathfrak{h},$$

with the following relations:

- $[\mathsf{h}, \mathsf{E}_i] = \alpha_i(\mathsf{h})\mathsf{E}_i$;
- $[\mathsf{h}, \mathsf{F}_i] = -\alpha_i(\mathsf{h})\mathsf{F}_i$;
- $[\mathsf{E}_i, \mathsf{F}_j] = \delta_{ij} \frac{\mathsf{q}^{d_i \mathsf{h}_i} - \mathsf{q}^{-d_i \mathsf{h}_i}}{\mathsf{q}^{d_i} - \mathsf{q}^{-d_i}}$;
- *and the* q*-Serre relations.*

Here, we understand everything involving q in terms of the Taylor expansions: for example, $\mathsf{q}^{d_i \mathsf{h}_i} = \sum_{n=0}^{\infty} d_i^n \mathsf{h}_i^n \hbar^n / 2^n n!$.

Remark 4.32

$$\mathfrak{U}_{\mathsf{q}}(\mathfrak{g}) / \hbar \mathfrak{U}_{\mathsf{q}}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}),$$

since factoring out $\hbar \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ is equivalent to setting $\hbar = 0$, or $\mathsf{q} = 1$.

Theorem 4.33 $\mathfrak{U}_{\mathsf{q}}$ *is a topologically free* $\mathbf{C}[[\hbar]]$*-module; that is, there exists an isomorphism of* $\mathbf{C}[[\hbar]]$*-modules*

$$\phi : \mathfrak{U}_{\mathsf{q}}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})[[\hbar]]$$

such that $\phi \bmod \hbar = \mathsf{Id}$.

A stronger theorem is the following.

Theorem 4.34 (Drinfeld (1990*b*)) ϕ *can be arranged to be an algebra isomorphism.*

This means that $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{g})[[\hbar]]$ have the same algebra structure, which explains why the representation theory of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ is essentially the same as that of $\mathfrak{U}(\mathfrak{g})$. However, $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{g})[[\hbar]]$ have different *Hopf algebra* structures (namely, the comultiplications are different), and this is what makes $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ interesting.

4.9 Quasi-triangular Hopf algebras

Definition 4.35 *A quasi-triangular structure on a Hopf algebra* $\mathbf{H}$ *is an invertible element* $R \in \mathbf{H} \otimes \mathbf{H}$ *such that:*

1. *For all* $x \in \mathbf{H}$, $R\mathbf{\Delta}(x) = \mathbf{\Delta}^{\mathrm{op}}(x)R$.
2. *The following hexagon relations are satisfied (in* $\mathbf{H} \otimes \mathbf{H} \otimes \mathbf{H}$*):*
   - *(a)* $(\mathbf{\Delta} \otimes \mathsf{Id})(R) = R^{13}R^{23}$,
   - *(b)* $(\mathsf{Id} \otimes \mathbf{\Delta})(R) = R^{13}R^{12}$.

A Hopf algebra $\mathbf{H}$ *equipped with a quasi-triangular structure is said to be a quasi-triangular Hopf algebra.*

Lemma 4.36 *If* R *is a quasi-triangular structure on* $\mathbf{H}$*, then so is* $(R^{21})^{-1}$.

Proof For (1), we have, for all $x \in \mathbf{H}$:

$$\begin{aligned} R^{12}\boldsymbol{\Delta}(x) &= \boldsymbol{\Delta}^{21}(x)R^{12} \quad \text{since axiom (1) holds for } R \\ \Longrightarrow R^{21}\boldsymbol{\Delta}^{21}(x) &= \boldsymbol{\Delta}(x)R^{21} \\ \Longrightarrow (R^{21})^{-1}\boldsymbol{\Delta}(x) &= \boldsymbol{\Delta}^{21}(x)(R^{21})^{-1}, \end{aligned}$$

thus axiom (1) holds for $(R^{21})^{-1}$. For axiom (2a), we let $R' = (R^{21})^{-1}$. Then we have:

$$\begin{aligned} (\mathsf{Id} \otimes \boldsymbol{\Delta})R &= R^{13}R^{12} \quad \text{since axiom (2b) holds for } R \\ \Longrightarrow (\boldsymbol{\Delta} \otimes \mathsf{Id})R^{21} &= R^{32}R^{31} \\ \Longrightarrow (\boldsymbol{\Delta} \otimes \mathsf{Id})(R')^{-1} &= ((R')^{23})^{-1}((R')^{13})^{-1} \\ \Longrightarrow (\boldsymbol{\Delta} \otimes \mathsf{Id})R' &= (R')^{13}(R')^{23}, \end{aligned}$$

thus axiom (2a) holds for $R' = (R^{21})^{-1}$. Similarly, axiom (2b) holds for $(R^{21})^{-1}$. □

Definition 4.37 *A triangular structure on a Hopf algebra* $\mathbf{H}$ *is a quasi-triangular structure* $R \in \mathbf{H} \otimes \mathbf{H}$ *such that* $(R^{21})^{-1} = R$. *A Hopf algebra* $\mathbf{H}$ *equipped with a triangular structure is said to be a triangular Hopf algebra.*

Remark 4.38 If $(R^{21})^{-1} = R$, then axioms (2a) and (2b) become equivalent.

Proposition 4.39 *If* $(\mathbf{H}, R)$ *is a quasi-triangular Hopf algebra, then* R *satisfies the quantum Yang–Baxter equation:*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$

Proof We have

$$\begin{aligned} R^{12}R^{13}R^{23} &= R^{12}(\boldsymbol{\Delta} \otimes \mathsf{Id})(R) \quad \text{by axiom (2a)} \\ &= (\boldsymbol{\Delta}^{\text{op}} \otimes \mathsf{Id})(R)R^{12} \quad \text{by (1)} \\ &= (R^{13}R^{23})^{1\leftrightarrow 2}R^{12} \quad \text{by axiom (2a)} \\ &= R^{23}R^{13}R^{12}. \end{aligned}$$

□

Lemma 4.40 *Let* $(\mathbf{H}, R)$ *be a quasi-triangular Hopf algebra. Then,*

$$(\epsilon \otimes \mathsf{Id})(R) = (\mathsf{Id} \otimes \epsilon)(R) = 1.$$

Proof We have

$$\begin{aligned} R &= (\epsilon \otimes \mathsf{Id} \otimes \mathsf{Id})(\mathbf{\Delta} \otimes \mathsf{Id})(R) \\ &= (\epsilon \otimes \mathsf{Id} \otimes \mathsf{Id})(R^{13}R^{23}) \quad \text{since } R \text{ is a quasi-triangular structure} \\ &= (\epsilon \otimes \mathsf{Id})(R) \cdot R. \end{aligned}$$

Since R is invertible, it follows that $(\epsilon \otimes \mathsf{Id})(R) = 1$. Similarly, $(\mathsf{Id} \otimes \epsilon)(R) = 1$.

□

The following proposition will be useful later.

Proposition 4.41 *Let* $(\mathbf{H}, R)$ *be a quasi-triangular Hopf algebra. Then,*

1. $(\mathbf{S} \otimes \mathsf{Id})(R) = (\mathsf{Id} \otimes \mathbf{S})(R) = R^{-1}$;
2. $(\mathbf{S} \otimes \mathsf{Id})(R^{-1}) = (\mathsf{Id} \otimes \mathbf{S})(R^{-1}) = R$;
3. $(\mathbf{S} \otimes \mathbf{S})(R) = R$.

Proof

1. Since R is quasi-triangular, we have $(\mathbf{\Delta} \otimes \mathsf{Id})(R) = R^{13}R^{23}$. We now apply $\mathbf{m}_{12}(\mathbf{S} \otimes \mathsf{Id} \otimes \mathsf{Id})$ to both sides:

$$\begin{aligned} (\mathbf{m}_{12} \otimes \mathsf{Id}_3)(\mathbf{S} \otimes \mathsf{Id} \otimes \mathsf{Id})(\mathbf{\Delta} \otimes \mathsf{Id})(R) &= (i \circ \epsilon \otimes \mathsf{Id})(R) \\ &= 1 \quad \text{by Lemma 4.40;} \\ (\mathbf{m}_{12} \otimes \mathsf{Id}_3)(\mathbf{S} \otimes \mathsf{Id} \otimes \mathsf{Id})(R^{13}R^{23}) &= (\mathbf{S} \otimes \mathsf{Id})(R) \cdot R. \end{aligned}$$

Therefore, $1 = (\mathbf{S} \otimes \mathsf{Id})(R) \cdot R$, and hence $(\mathbf{S} \otimes \mathsf{Id})(R) = R^{-1}$. Similarly, $(\mathsf{Id} \otimes \mathbf{S})(R) = R^{-1}$.
2. Since $(R^{21})^{-1}$ is also a quasi-triangular structure for $\mathbf{H}$, we see that

$$(\mathsf{Id} \otimes \mathbf{S})\left((R^{21})^{-1}\right) = R^{21};$$

hence, $(\mathbf{S} \otimes \mathsf{Id})(R^{-1}) = R$. Similarly, $(\mathsf{Id} \otimes \mathbf{S})(R^{-1}) = R$.
3. We see that

$$(\mathbf{S} \otimes \mathbf{S})(R) = (\mathsf{Id} \otimes \mathbf{S})(\mathbf{S} \otimes \mathsf{Id})(R) = (\mathsf{Id} \otimes \mathbf{S})(R^{-1}) = R.$$

□

4.10 Quasi-triangular Hopf algebras and representation theory

Let $\mathbf{H}$ be a Hopf algebra, and let $R \in \mathbf{H} \otimes \mathbf{H}$ be a quasi-triangular structure. If V, W are representations of $\mathbf{H}$, then we can define

$$R_{VW} = (\pi_V \otimes \pi_W)R : V \otimes W \to V \otimes W \text{ and } \check{R}_{VW} = PR_{VW} : V \otimes W \to W \otimes V.$$

Remark 4.42 For any representation V of $\mathbf{H}$, $R_{VV} : V \otimes V \to V \otimes V$ satisfies the quantum Yang–Baxter equation.

Proposition 4.43 *$\check{R}_{VW}$ is an isomorphism of representations of $\mathbf{H}$.*

Proof We first check that $\check{R}_{VW}$ is a homomorphism of representations of $\mathbf{H}$ (that is, a morphism in $\mathsf{Rep}\,\mathbf{H}$). It is enough to show that $\check{R}_{VW}$ commutes with $\mathbf{\Delta}(x)$ for all x; and indeed we have

$$\begin{aligned} \check{R}_{VW}\mathbf{\Delta}(x) &= PR\mathbf{\Delta}(x) \\ &= P\mathbf{\Delta}^{\mathrm{op}}(x)R \quad \text{since } R \text{ satisfies axiom (1)} \\ &= \mathbf{\Delta}(x)PR. \end{aligned}$$

Since R is invertible, $\check{R}_{VW}$ is actually an isomorphism of representations. □

Proposition 4.44 (Functorial property) *Let $\phi : V' \to V, \psi : W' \to W$ be homomorphisms. Then, the following diagram commutes:*

$$\begin{array}{ccc} V' \otimes W' & \xrightarrow{\check{R}_{V'W'}} & W' \otimes V' \\ {\scriptstyle \phi\otimes\psi}\downarrow & & \downarrow{\scriptstyle \psi\otimes\phi} \\ V \otimes W & \xrightarrow{\check{R}_{VW}} & W \otimes V \end{array} \tag{4.4}$$

Proposition 4.45 *Suppose V_1, V_2, V_3, V and W are in $\mathsf{Rep}\,\mathbf{H}$.*

1. *Axiom* (2a) *is equivalent to the commutativity of the following diagram:*

$$\begin{array}{ccc} V_1 \otimes V_2 \otimes W & \xrightarrow{\check{R}_{V_1\otimes V_2, W}} & W \otimes V_1 \otimes V_2 \, . \\ {\scriptstyle \mathsf{Id}\otimes\check{R}_{V_2W}}\downarrow & \nearrow{\scriptstyle \check{R}_{V_1W}\otimes\mathsf{Id}} & \\ V_1 \otimes W \otimes V_2 & & \end{array} \tag{4.5}$$

2. *Axiom* (2b) *is equivalent to the commutativity of the following diagram:*

$$\begin{array}{ccc} W \otimes V_1 \otimes V_2 & \xrightarrow{\check{R}_{W, V_1\otimes V_2}} & V_1 \otimes V_2 \otimes W \, . \\ {\scriptstyle \check{R}_{WV_1}\otimes\mathsf{Id}}\downarrow & \nearrow{\scriptstyle \mathsf{Id}\otimes\check{R}_{WV_2}} & \\ V_1 \otimes W \otimes V_2 & & \end{array} \tag{4.6}$$

3. *The triangular property $(R^{21})^{-1} = R$ is equivalent to the commutativity of the following diagram:*

$$\begin{array}{ccc} V \otimes W & \xrightarrow{\mathsf{Id}} & V \otimes W \\ {\scriptstyle \check{R}_{VW}}\downarrow & \nearrow {\scriptstyle \check{R}_{WV}} & \\ W \otimes V & & \end{array} \quad . \tag{4.7}$$

4. *The quantum Yang–Baxter equation is equivalent to the commutativity of the following diagram:*

$$\begin{array}{ccccc} & & V_1 \otimes V_2 \otimes V_3 & & \\ & {\scriptstyle \check{R}_{V_1V_2}\otimes \mathsf{Id}} \swarrow & & \searrow {\scriptstyle \mathsf{Id}\otimes \check{R}_{V_2V_3}} & \\ V_2 \otimes V_1 \otimes V_3 & & & & V_1 \otimes V_3 \otimes V_2 \\ {\scriptstyle \mathsf{Id}\otimes \check{R}_{V_1V_3}}\downarrow & & & & \downarrow {\scriptstyle \check{R}_{V_1V_3}\otimes \mathsf{Id}} \\ V_2 \otimes V_3 \otimes V_1 & & & & V_3 \otimes V_1 \otimes V_2 \\ & {\scriptstyle \check{R}_{V_2V_3}\otimes \mathsf{Id}} \searrow & & \swarrow {\scriptstyle \mathsf{Id}\otimes \check{R}_{V_1V_2}} & \\ & & V_3 \otimes V_2 \otimes V_1 & & \end{array} \quad . \tag{4.8}$$

Recall (Bakalov and Kirillov (2001)) that a tensor category (over $\mathbf{C}$) is a $\mathbf{C}$-linear additive category $\mathcal{C}$ with a functor of tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and associativity isomorphism $\phi : (\bullet \otimes \bullet) \otimes \bullet \to \bullet \otimes (\bullet \otimes \bullet)$ satisfying some axioms (most importantly, the pentagon identity for ϕ).

Definition 4.46 *A functorial morphism $\check{R} : V \otimes W \to W \otimes V$ in a tensor category $\mathcal{C}$ is said to be a braided structure or braiding if it makes diagrams* (4.5) *and* (4.6) *commute. A braided structure is said to be a symmetric structure if $\check{R}^2 = \mathsf{Id}$.*

Corollary 4.47

1. *If $(\mathbf{H}, R)$ is a quasi-triangular Hopf algebra, then* $\mathsf{Rep}\,\mathbf{H}$ *is a braided tensor category.*
2. *If $(\mathbf{H}, R)$ is a triangular Hopf algebra, then* $\mathsf{Rep}\,\mathbf{H}$ *is a symmetric tensor category.*

Definition 4.48 *The braid group $\mathfrak{B}_n$ is the group with generators $b_1, \dots, b_{n-1}$ with relations $b_i b_j = b_j b_i, |i - j| \geq 2$ and $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, i = 1, \dots n - 2$.*

Remark 4.49 We have $\mathfrak{S}_n = \mathfrak{B}_n / \langle b_i^2 = 1 \rangle$, where $\mathfrak{S}_n$ denotes the symmetric group of permutations of n items.

Proposition 4.50

1. *Let* $(\mathcal{C}, \check{R})$ *be a braided tensor category, and let* $V \in \mathcal{C}$*. Then, for all* $n \geq 2$*, there is a homomorphism* $\phi_n : \mathfrak{B}_n \to \mathsf{Aut}(\underbrace{V \otimes \cdots \otimes V}_{n})$ *given by*

   $\phi_n(b_i) = (\check{R}_{VV})^{i,i+1}$.
2. ϕ_n *factors through* $\mathfrak{S}_n$ *for all* V *(i.e.* $\phi_n = \mathsf{Id}$ *on* $\mathsf{ker}(\mathfrak{B}_n \to \mathfrak{S}_n)$*) if and only if the braided structure is symmetric.*

Not all Hopf algebras admit a quasi-triangular structure.

Example 4.51 If G is a non-abelian group, then $\mathbf{H} = F(G)$ is *not* quasi-triangular. To see this, we note that if R is a quasi-triangular structure, then $\check{R}$ induces an isomorphism $V \otimes W \cong W \otimes V$, for all $V, W \in \mathsf{Rep}\,\mathbf{H}$. Now, for each $g \in G$, let χ_g be the representation of $\mathbf{H}$ such that $\chi_g(f) = f(g)$ for all $f \in F(G)$. Let $g_1, g_2 \in G$ be such that $g_1 g_2 \neq g_2 g_1$. Then, $\chi_{g_1} \otimes \chi_{g_2} = \chi_{g_1 g_2} \ncong \chi_{g_2 g_1} = \chi_{g_2} \otimes \chi_{g_1}$.

However, $\mathbf{H}^* = \mathbf{C}[G]$ is quasi-triangular, with $R = 1_G \otimes 1_G$. This is an instance of a more general fact: a Hopf algebra $\mathbf{H}$ is cocommutative (that is, $\boldsymbol{\Delta} = \boldsymbol{\Delta}^{\mathrm{op}}$) if and only if $R = 1 \otimes 1$ is a quasi-triangular structure.

4.11 Quasi-triangularity and $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$

For the moment, let us think of q as $e^{\hbar/2}$, where $\hbar$ is a formal parameter. We should now ask ourselves: does the Hopf algebra $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$ admit a quasi-triangular structure? Strictly speaking, the answer to this question is *no*. However, we will see that there exists an $\mathcal{R}$ which satisfies the axioms for a quasi-triangular structure in a *completion* of $\mathfrak{U}_\mathsf{q}(\mathfrak{g}) \otimes \mathfrak{U}_\mathsf{q}(\mathfrak{g})$:

Definition 4.52 *Let* $V[[\hbar]], W[[\hbar]]$ *be topologically free* $\mathbf{C}[[\hbar]]$*-modules. Then define*

$$V[[\hbar]] \hat{\otimes} W[[\hbar]] \stackrel{\mathrm{def}}{=} (V \otimes W)[[\hbar]].$$

Remark 4.53 $V[[\hbar]] \hat{\otimes} W[[\hbar]]$ is not the same as $V[[\hbar]] \otimes W[[\hbar]]$ when V, W are infinite-dimensional. Indeed, if $\{v_i, i \in \mathbf{Z}_+\}$ and $\{w_i, i \in \mathbf{Z}_+\}$ are linearly independent sets in V and W respectively, then $\sum_i \hbar^i v_i \otimes w_i$ is in $V[[\hbar]] \hat{\otimes} W[[\hbar]]$ but not in $V[[\hbar]] \otimes W[[\hbar]]$.

Theorem 4.54 *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra. Then there exists $\mathcal{R} \in \mathfrak{U}_{\mathsf{q}}(\mathfrak{g}) \hat{\otimes} \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ which satisfies the axioms for a quasi-triangular structure (but not the axiom for a triangular structure).*

Remark 4.55 We emphasize that $\mathcal{R}$ is an element of the completed tensor product $\mathfrak{U}_{\mathsf{q}}(\mathfrak{b}_+) \hat{\otimes} \mathfrak{U}_{\mathsf{q}}(\mathfrak{b}_-)$, but not of the usual tensor product $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g}) \otimes \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$.

Example 4.56 Let $\mathfrak{g} = \mathfrak{sl}_2$. Then the "quasi-triangular structure" on $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ is given by

$$\mathcal{R} = \mathsf{q}^{\frac{1}{2}\mathsf{h}\otimes\mathsf{h}} \sum_{n=0}^{\infty} \mathsf{q}^{\frac{n(n-1)}{2}} \frac{(\mathsf{q}-\mathsf{q}^{-1})^n}{[n]_{\mathsf{q}}!} (\mathsf{E}^n \otimes \mathsf{F}^n).$$

Let us now move from the formal setting to the numerical one; i.e., $\mathsf{q} \in \mathbf{C}^*, \mathsf{q}$ is not a root of unity.

Definition 4.57 *Let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra. A $\mathfrak{g}$-module (or a $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$-module) V is in category $\mathcal{O}$ if it is finitely generated, has a weight decomposition and $\mathfrak{U}_{\mathsf{q}}(\mathfrak{n}_+)v$ is finite-dimensional for all $v \in V$.*

Example 4.58 Again, let $\mathfrak{g} = \mathfrak{sl}_2$. Let V, W be representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$ in category $\mathcal{O}$. Then $\mathcal{R}$ from Example 4.56 defines a braiding on category $\mathcal{O}$. This is because for every $v \otimes w \in V \otimes W$, all but finitely many of the terms in the infinite sum will act as zero on $v \otimes w$ (since V is in category $\mathcal{O}$.).

Now, let us consider the case where $V = W = \langle v_+, v_- \rangle$, the two-dimensional representation with action $\mathsf{E}v_- = v_+, \mathsf{E}v_+ = 0$, $\mathsf{F}v_- = 0, \mathsf{F}v_+ = v_-$, $\mathsf{h}v_+ = v_+, \mathsf{h}v_- = -v_-$. Then we can compute

$$[\mathcal{R}|_{V\otimes V}] = \begin{pmatrix} \mathsf{q}^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \mathsf{q}^{-\frac{1}{2}} & \mathsf{q}^{-\frac{1}{2}}(\mathsf{q}-\mathsf{q}^{-1}) & 0 \\ 0 & 0 & \mathsf{q}^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \mathsf{q}^{\frac{1}{2}} \end{pmatrix},$$

where the ordered basis of $V \otimes V$ is

$$(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-).$$

This is a solution of the quantum Yang–Baxter equation.

Remark 4.59 In Example 4.58, if we set $\mathsf{q} = 1$, then $[\mathcal{R}|_{V\otimes V}]$ becomes the identity. This illustrates the fact that from $\mathfrak{U}(\mathfrak{g})$, we cannot obtain any interesting solutions of the quantum Yang–Baxter equation.

Remark 4.60 Note that the category $\mathsf{Rep}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ of all representations of the quantum group is *not* braided, (as $\mathcal{R}$ lies in the completed and not usual tensor product), but the category $\mathcal{O}$ is braided.

Remark 4.61 In Lie algebra theory, we generally normalize our inner product so that $\langle\theta,\theta\rangle = 2$. But in quantum group theory, we normalize so that $\langle\alpha_{sh},\alpha_{sh}\rangle = 2$, where α_{sh} is a short root. The two are equal only when $\mathfrak{g}$ is simply laced.

Theorem 4.62 *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\{\mathsf{x}_i\}$ an orthonormal basis of $\mathfrak{h}$ (with respect to the form of remark 4.61). Then there exists a unique*

$$\mathcal{R} = \mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\left(1 + \sum_{\beta\in\mathsf{Q}_+\setminus\{0\}} L^{(\beta)}\right) \quad \textit{with } L^{(\beta)} \in \mathfrak{U}_{\mathsf{q}}(\mathfrak{b}_+)[\beta]\otimes\mathfrak{U}_{\mathsf{q}}(\mathfrak{b}_-)[-\beta],$$

which satisfies the quasi-triangularity axioms, after being evaluated in representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ for category $\mathcal{O}$.

Remark 4.63 This element $\mathcal{R}$ coincides with the one from Theorem 4.54 upon expansion in powers of $\hbar$.

Example 4.64 If $\mathfrak{g}$ is simply laced, then $L^{(\alpha_i)} = (\mathsf{q}-\mathsf{q}^{-1})\mathsf{E}_i\otimes\mathsf{F}_i$ for any simple root α_i.

Definition 4.65 *The quasi-triangular structure $\mathcal{R}$ is known as the universal R-matrix.*

4.12 Twisting

Let $(\mathbf{H},\mathbf{m},i,\boldsymbol{\Delta},\epsilon,\mathbf{S})$ be a Hopf algebra.

Definition 4.66 (Drinfeld) *Let $J\in\mathbf{H}\otimes\mathbf{H}$ be an invertible element. Then J is called a twist if*

$$((\boldsymbol{\Delta}\otimes\mathsf{Id})(J))(J\otimes 1) = ((\mathsf{Id}\otimes\boldsymbol{\Delta})(J))(1\otimes J) \quad \textit{in } \mathbf{H}\otimes\mathbf{H}\otimes\mathbf{H},$$

or, equivalently,

$$J^{12,3}J^{1,2} = J^{1,23}J^{2,3}. \tag{4.9}$$

Remark 4.67 Equation (4.9) is a non-dynamical version of the dynamical twist equation.

Proposition 4.68 *Let $\mathbf{\Delta}_J : \mathbf{H} \to \mathbf{H} \otimes \mathbf{H}$ be given by*

$$\mathbf{\Delta}_J(x) \stackrel{\text{def}}{=} J^{-1}\mathbf{\Delta}(x)J.$$

Then there exists $\mathbf{S}_J : \mathbf{H} \to \mathbf{H}$ such that $(\mathbf{H}, \mathbf{m}, i, \mathbf{\Delta}_J, \epsilon, \mathbf{S}_J)$ is a Hopf algebra.

Definition 4.69 *The Hopf algebra in Proposition 4.68 is denoted $\mathbf{H}^J$ and is called the twist of $\mathbf{H}$ by J.*

Proposition 4.70 *If $(\mathbf{H}, R)$ is a quasi-triangular Hopf algebra with twist J, then $(\mathbf{H}^J, R^J)$, where $R^J = (J^{21})^{-1}RJ$, is also quasi-triangular.*

Proposition 4.71 *If J is a twist and $x \in \mathbf{H}$ is invertible, then*

$$\mathbf{\Delta}(x)J(x^{-1} \otimes x^{-1})$$

is also a twist.

Example 4.72 If we take $J = 1 \otimes 1$, then for all invertible $x \in \mathbf{H}$,

$$\mathbf{\Delta}(x)(x^{-1} \otimes x^{-1})$$

is a twist by Proposition 4.71.

Definition 4.73 *We say that J_1 and J_2 are gauge equivalent if*

$$J_2 = \mathbf{\Delta}(x)J_1(x^{-1} \otimes x^{-1})$$

for some invertible $x \in \mathbf{H}$.

Proposition 4.74 *If J_1 and J_2 are gauge equivalent, then $\mathbf{H}^{J_1} \cong \mathbf{H}^{J_2}$.*

Example 4.75 If $(\mathbf{H}, R)$ is quasitriangular, then R^{-1} is a twist. The associated comultiplication is $\mathbf{\Delta}_{R^{-1}}(x) = R\mathbf{\Delta}(x)R^{-1} = \mathbf{\Delta}^{\text{op}}(x)$, and the twist of $\mathbf{H}$ by R^{-1} is $\mathbf{H}^{op} = (\mathbf{H}, \mathbf{m}, i, \mathbf{\Delta}^{\text{op}}, \epsilon, \mathbf{S}^{-1})$

Example 4.76 Let $\mathfrak{g}$ be a semisimple Lie algebra, and let α_i, α_j be long simple roots such that $\langle \alpha_i, \alpha_j \rangle = 0$. We would like to construct a "twist" in $\mathfrak{U}_\mathsf{q}(\mathfrak{g})\hat{\otimes}\mathfrak{U}_\mathsf{q}(\mathfrak{g})$. We note that the Hopf algebra $\langle \mathsf{h}_i, \mathsf{E}_i \rangle$ is isomorphic to $\mathfrak{U}_\mathsf{q}(\mathfrak{b}_+^{\mathfrak{sl}_2})$, and the Hopf algebra $\langle \mathsf{h}_j, \mathsf{F}_j \rangle$ is isomorphic to $\mathfrak{U}_\mathsf{q}(\mathfrak{b}_-^{\mathfrak{sl}_2})$. So there exists a natural embedding Ψ of $\mathfrak{U}_\mathsf{q}(\mathfrak{b}_+^{\mathfrak{sl}_2})\hat{\otimes}\mathfrak{U}_\mathsf{q}(\mathfrak{b}_-^{\mathfrak{sl}_2})$ into $\langle \mathsf{h}_i, \mathsf{E}_i \rangle \hat{\otimes} \langle \mathsf{h}_j, \mathsf{F}_j \rangle \subset \mathfrak{U}_\mathsf{q}(\mathfrak{g})\hat{\otimes}\mathfrak{U}_\mathsf{q}(\mathfrak{g})$. Let J be the image under Ψ of the "quasi-triangular structure" $\mathcal{R} \in \mathfrak{U}_\mathsf{q}(\mathfrak{b}_+^{\mathfrak{sl}_2})\hat{\otimes}\mathfrak{U}_\mathsf{q}(\mathfrak{b}_-^{\mathfrak{sl}_2})$. Then J satisfies the properties of a twist.

Proof We know that $(\mathbf{\Delta} \otimes \mathsf{Id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}$ and $(\mathsf{Id} \otimes \mathbf{\Delta})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}$. Thus $(\mathbf{\Delta} \otimes \mathsf{Id})(J)J^{12} = J^{13}J^{23}J^{12}$ and $(\mathsf{Id} \otimes \mathbf{\Delta})(J)J^{23} = J^{13}J^{12}J^{23}$. But $\langle \alpha_i, \alpha_j \rangle = 0$, so the algebras $\langle \mathsf{E}_i, \mathsf{h}_i \rangle$ and $\langle \mathsf{F}_j, \mathsf{h}_j \rangle$ commute, which implies that $J^{12}J^{23} = J^{23}J^{12}$. Therefore,

$$(\mathbf{\Delta} \otimes \mathsf{Id})(J)J^{12} = J^{13}J^{23}J^{12} = J^{13}J^{12}J^{23} = (\mathsf{Id} \otimes \mathbf{\Delta})(J)J^{23},$$

so indeed J satisfies the properties of a twist. □

4.13 Quasi-classical limit for the quantum Yang–Baxter equation

Let $\mathfrak{g}$ be a semisimple Lie algebra. In Section 4.11, we saw that there exists a universal R-matrix $\mathcal{R} \in \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})\hat{\otimes}\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ satisfying the quasi-triangularity axioms. This $\mathcal{R}$ satisfies $\lim_{\mathsf{q}\to 1} \mathcal{R} = 1$. so that $\mathcal{R} = 1 + \hbar r + O(\hbar^2)$ for some $r \in \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$. In this section, we will study r.

Proposition 4.77 *Let A be an associative algebra. Let $R \in (A \otimes A)[[\hbar]]$ satisfy the quantum Yang–Baxter equation,*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{4.10}$$

Write $R = 1 + \hbar r + O(\hbar^2)$, where $r \in A \otimes A$. Then

$$\left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right] = 0. \tag{4.11}$$

Equation (4.11) is known as the *classical Yang–Baxter equation.*

Proof By substituting $R = 1 + \hbar r + \hbar^2 s + O(\hbar^3)$ in (4.10) (here $s \in A \otimes A$) and extracting the coefficient of $\hbar^2$ from both sides, we obtain

$$s^{12}+s^{13}+s^{23}+r^{12}r^{13}+r^{12}r^{23}+r^{13}r^{23} = s^{23}+s^{13}+s^{12}+r^{23}r^{13}+r^{23}r^{12}+r^{13}r^{12},$$

whence the result follows. □

r is called the *quasi-classical limit* of R, and R is a *quantization* of r.

A natural question to ask is whether every solution r of (4.11) admits a quantization. The answer is provided by the following.

Theorem 4.78 (Etingof and Kazhdan (1996)) *Every solution r of* (4.11) *admits a quantization.*

Remark 4.79 Given a solution r of (4.11), a quantization of r is not unique. For example, if $r = 0$, then $R = 1$ is a quantization, but so is $R(\hbar^2)$, where $R(\hbar) = 1 + O(\hbar)$ is any solution of (4.10).

Example 4.80 Consider $\mathfrak{U}_q(\mathfrak{sl}_2)$. We know that

$$\mathcal{R} = q^{\frac{1}{2}h\otimes h}\left(1 + (q - q^{-1})E \otimes F + q\frac{(q - q^{-1})^2}{[2]_q!}E^2 \otimes F^2 + \cdots\right)$$

satisfies the axioms of quasi-triangularity. Recall that $q = e^{\frac{\hbar}{2}}$, and denote $e = \lim_{q\to 1} E, f = \lim_{q\to 1} F$. Then

$$\mathcal{R} = 1 + \hbar\left(\tfrac{1}{4}h \otimes h + e \otimes f\right) + O(\hbar^2).$$

Thus

$$r = \tfrac{1}{4}h \otimes h + e \otimes f \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \qquad (4.12)$$

is the quasi-classical limit of $\mathcal{R}$ and hence satisfies (4.11).

More generally, if $\mathfrak{g}$ is an arbitrary semisimple Lie algebra and $\mathcal{R}$ is the universal R-matrix for $\mathfrak{U}_q(\mathfrak{g})$, then the quasi-classical limit of $\mathcal{R}$ is

$$r = \frac{1}{2}\sum_i x_i \otimes x_i + \sum_{\alpha\in R_+} e_\alpha \otimes f_\alpha \in \mathfrak{g} \otimes \mathfrak{g}. \qquad (4.13)$$

This r also satisfies (4.11).

Equation (4.13) is known as Drinfeld's solution to the classical Yang–Baxter equation.

4.14 Quasi-classical limit for quantum dynamical Yang–Baxter equation

Let $\mathfrak{g}$ be a semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and $\{x_i\}$ an orthonormal basis of $\mathfrak{h}$. Recall the quantum dynamical Yang–Baxter equation (3.2). A solution of (3.2) is $R(\lambda) : V \otimes V \to V \otimes V$, where V is a diagonalizable $\mathfrak{h}$-module. We would like to find the quasi-classical analog of (3.2). Introduce a formal parameter $\hbar$, and make the change of variable $\lambda \to \lambda/\hbar$. Then (3.2) becomes

$$R^{12}(\lambda - \hbar h^3)R^{13}(\lambda)R^{23}(\lambda - \hbar h^1) = R^{23}(\lambda)R^{13}(\lambda - \hbar h^2)R^{12}(\lambda). \qquad (4.14)$$

Suppose $R(\lambda, \hbar)$ is a solution of (4.14), of the form

$$R(\lambda, \hbar) = 1 - r(\lambda)\hbar + s(\lambda)\hbar^2 + O(\hbar^3). \qquad (4.15)$$

We note that, for all $v_1, v_2, v_3 \in V$

$$\begin{aligned} &r^{12}(\lambda - \hbar h^3)(v_1 \otimes v_2 \otimes v_3) \\ &\quad = r^{12}(\lambda - \hbar\, \mathrm{wt}\, v_3)(v_1 \otimes v_2) \otimes v_3 \\ &\quad = \left(r^{12}(\lambda) - \hbar\sum_i \frac{\partial r^{12}(\lambda)}{\partial x_i}\langle \mathrm{wt}\, v_3, x_i\rangle + O(\hbar^2)\right)(v_1 \otimes v_2 \otimes v_3) \\ &\quad = \left(r^{12}(\lambda) - \hbar\sum_i \frac{\partial r^{12}(\lambda)}{\partial x_i} \otimes x_i + O(\hbar^2)\right)(v_1 \otimes v_2 \otimes v_3). \qquad (4.16) \end{aligned}$$

(Here the second equality arises from the Taylor series expansion.)

Substituting (4.15) and (4.16) (as well as the analogs of (4.16) for $r^{13}(\lambda-\hbar h^2)$ and $r^{23}(\lambda-\hbar h^1)$) in (4.14) and extracting the coefficient of $\hbar^2$ gives

$$\sum_i \left(\frac{\partial r^{12}}{\partial \mathsf{x}_i}\mathsf{x}_i^3 - \frac{\partial r^{13}}{\partial \mathsf{x}_i}\mathsf{x}_i^2 + \frac{\partial r^{23}}{\partial \mathsf{x}_i}\mathsf{x}_i^1\right) + \left[r^{12}, r^{13}\right] + \left[r^{12}, r^{23}\right] + \left[r^{13}, r^{23}\right] = 0. \quad (4.17)$$

Equation (4.17) is known as the *classical dynamical Yang–Baxter equation.*

Definition 4.81 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ an abelian subalgebra. A classical dynamical r-matrix is a meromorphic function $r : \mathfrak{h}^* \to (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{h}}$ which satisfies the classical dynamical Yang–Baxter equation* (4.17).

Theorem 4.82 *Let $\mathfrak{g}$ be a semisimple Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. Let $R(\lambda) : V \otimes V \to V \otimes V$ be the exchange matrix of $\mathfrak{g}$. Then $R(\lambda/\hbar)$ admits a Taylor expansion in $\hbar$, and $R(\frac{\lambda}{\hbar}) = 1 - r(\lambda)\hbar + O(\hbar^2)$, where*

$$r(\lambda) = \sum_{\alpha \in \mathsf{R}_+} \frac{\mathsf{e}_\alpha \wedge \mathsf{e}_{-\alpha}}{\langle \lambda, \alpha \rangle} \in \Lambda^2 \mathfrak{g}.$$

Proof Recall, from Section 3.5, the ABRR equation, (3.13), of which $J(\lambda)$ is the unique upper triangular solution. Substituting $\lambda/\hbar$ for λ in (3.13) gives

$$\left[\tilde{J}, \mathsf{Id} \otimes \left(\frac{\bar{\lambda}}{\hbar} + \bar{\rho} - \frac{1}{2}\sum_i \mathsf{x}_i^2\right)\right] = \sum_{\alpha \in \mathsf{R}_+} (\mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha)\tilde{J}, \quad (4.18)$$

where $\tilde{J}(\lambda) = J(\lambda/\hbar) = 1 + j(\lambda)\hbar + O(\hbar^2)$. Extracting the coefficient of $\hbar^0$ from both sides of (4.18) gives

$$\left[j(\lambda), \mathsf{Id} \otimes \bar{\lambda}\right] = \sum_{\alpha \in \mathsf{R}_+} \mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha. \quad (4.19)$$

But $j(\lambda)$ is of zero weight (since $J(\lambda)$ is), so we can write

$$j(\lambda) = \sum_{\beta > 0} j^\beta(\lambda), \quad \text{where } j^\beta(\lambda) \in (\mathsf{End}\, V)[-\beta] \otimes (\mathsf{End}\, V)[\beta].$$

Then (4.19) becomes

$$\sum_{\alpha \in \mathsf{R}_+} \mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha = -\sum_{\beta > 0} \langle \lambda, \beta \rangle\, j^\beta(\lambda),$$

so

$$j^{\beta}(\lambda) = \begin{cases} -\frac{\mathsf{e}_{-\beta}\otimes\mathsf{e}_{\beta}}{\langle\lambda,\beta\rangle} & \text{if } \beta\in\mathsf{R}_{+}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$j(\lambda) = -\sum_{\alpha\in\mathsf{R}_{+}} \frac{\mathsf{e}_{-\alpha}\otimes\mathsf{e}_{\alpha}}{\langle\lambda,\alpha\rangle}. \tag{4.20}$$

Now, we recall that $R(\lambda) = J(\lambda)^{-1}J^{21}(\lambda)$; thus,

$$1 - r(\lambda)\hbar + O(\hbar^2) = R(\frac{\lambda}{\hbar}) = \tilde{J}(\lambda)^{-1}\tilde{J}^{21}(\lambda) = 1 + \left(-j(\lambda) + j^{21}(\lambda)\right)\hbar + O(\hbar^2) \tag{4.21}$$

Combining (4.20) and (4.21) yields

$$r(\lambda) = j(\lambda) - j^{21}(\lambda) = \sum_{\alpha\in\mathsf{R}_{+}} \frac{-\mathsf{e}_{-\alpha}\otimes\mathsf{e}_{\alpha} + \mathsf{e}_{\alpha}\otimes\mathsf{e}_{-\alpha}}{\langle\lambda,\alpha\rangle} = \sum_{\alpha\in\mathsf{R}_{+}} \frac{\mathsf{e}_{\alpha}\wedge\mathsf{e}_{-\alpha}}{\langle\lambda,\alpha\rangle},$$

as required. □

Remark 4.83 The function $r(\lambda)$ is called the basic rational solution of the classical dynamical Yang-Baxter equation.

Remark 4.84 Equation (4.19) is called the classical ABRR equation.

5

INTERTWINERS, FUSION AND EXCHANGE OPERATORS FOR $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$

5.1 Fusion operator for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$

In this section, we will let q be a nonzero complex number which is not a root of unity. We could also let $\mathsf{q} = e^{\hbar/2}$, where $\hbar$ is a formal parameter. Let $\mathfrak{g}$ be a semisimple Lie algebra.

We can define intertwining operators and the fusion operator for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ in the same way as we did for $\mathfrak{g}$. The fusion operator $J_{WV}(\lambda) : W \otimes V \to W \otimes V$ will then satisfy the following.

Proposition 5.1

1. *$J_{WV}(\lambda)$ has zero weight: for any weight δ, $J_{WV}(\lambda)$ maps $(W \otimes V)[\delta]$ into itself.*
2. *$J_{WV}(\lambda)$ is lower triangular with respect to the weight decomposition, and has ones on its diagonal. That is, $J_{WV}(\lambda)(w \otimes v) = w \otimes v + \sum_i c_i \otimes b_i$, where $\mathsf{wt}\, c_i < \mathsf{wt}\, w, \mathsf{wt}\, b_i > \mathsf{wt}\, v$, for all homogeneous v, w. In particular, $J_{WV}(\lambda)$ is invertible whenever defined.*
3. *$J_{WV}(\lambda)$ is a rational function of $\mathsf{q}^{\langle \lambda, \alpha_i \rangle}$.*

Theorem 5.2 *Fusion operators satisfy the dynamical twist equation in $W \otimes V \otimes U$, where W, V, U are finite-dimensional representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$:*

$$J_{W \otimes V, U}(\lambda) J_{WV}(\lambda - h^3) = J_{W, V \otimes U}(\lambda) J_{VU}(\lambda).$$

Example 5.3 Let $\mathfrak{g} = \mathfrak{sl}_2$. Take V to be the two-dimensional irreducible representation of $\mathfrak{g}$. In Example 3.17, we found $J_{VV}(\lambda)$ to be

$$[J_{VV}(\lambda)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{\lambda+1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the ordered basis of $V \otimes V$ is

$$(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-).$$

We would like to compute $J_{VV}(\lambda)$ for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$, where V is the two-dimensional irreducible representation of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$.

As before, each of these four basis elements must be fixed by $J_{VV}(\lambda)$, except $v_+ \otimes v_-$. So we must compute $J_{VV}(\lambda)(v_+ \otimes v_-)$. We have

$$\Phi_\lambda^{v_-} \mathbf{v}_\lambda = \mathbf{v}_{\lambda+1} \otimes v_- + a(\lambda, \mathsf{q}) \mathsf{F} \mathbf{v}_{\lambda+1} \otimes v_+,$$

for some function a of λ and q. We then note that

$$\left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, \Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle = v_+,$$

and

$$\begin{aligned} &\left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, \Phi_{\lambda+1}^{v_+} \mathsf{F} \mathbf{v}_{\lambda+1} \right\rangle \\ &\quad = \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, (\mathsf{F} \otimes \mathsf{Id} + \mathsf{q}^{-\mathsf{h}} \otimes \mathsf{F}) \Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle \\ &\quad = \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, (\mathsf{q}^{-\mathsf{h}} \otimes \mathsf{F}) \Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle \quad \text{since } \langle \mathbf{v}_\lambda^*, \mathsf{F} w \rangle = 0 \text{ for all } w \in M_\lambda \\ &\quad = \mathsf{q}^{-\lambda} \mathsf{F} \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, \Phi_{\lambda+1}^{v_+} \mathbf{v}_{\lambda+1} \right\rangle \\ &\quad = \mathsf{q}^{-\lambda} \mathsf{F} v_+ = \mathsf{q}^{-\lambda} v_-. \end{aligned}$$

Therefore,

$$\begin{aligned} J_{VV}(\lambda)(v_+ \otimes v_-) &= \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, (\Phi_{\lambda+1}^{v_+} \otimes \mathsf{Id}) \Phi_\lambda^{v_-} \mathbf{v}_\lambda \right\rangle \\ &= \left\langle \mathbf{v}_\lambda^* \otimes \mathsf{Id}, (\Phi_{\lambda+1}^{v_+} \otimes \mathsf{Id})(\mathbf{v}_{\lambda+1} \otimes v_- + a(\lambda, \mathsf{q}) \mathsf{F} \mathbf{v}_{\lambda+1} \otimes v_+) \right\rangle \\ &= v_+ \otimes v_- + \mathsf{q}^{-\lambda} a(\lambda, \mathsf{q}) v_- \otimes v_+. \end{aligned}$$

Finally, we must determine $a(\lambda, \mathsf{q})$. We see that

$$\begin{aligned} 0 &= \Phi_\lambda^{v_-} \mathsf{E} \mathbf{v}_\lambda \\ &= \mathsf{E} \Phi_\lambda^{v_-} \mathbf{v}_\lambda \\ &= \mathsf{E}(\mathbf{v}_{\lambda+1} \otimes v_-) + a(\lambda, \mathsf{q}) \mathsf{E}(\mathsf{F} \mathbf{v}_{\lambda+1} \otimes v_+) \\ &= \mathsf{E} \mathbf{v}_{\lambda+1} \otimes \mathsf{q}^{\mathsf{h}} v_- + \mathbf{v}_{\lambda+1} \otimes \mathsf{E} v_- + a(\lambda, \mathsf{q})(\mathsf{EF} \mathbf{v}_{\lambda+1} \otimes \mathsf{q}^{\mathsf{h}} v_+ + \mathsf{F} \mathbf{v}_{\lambda+1} \otimes \mathsf{E} v_+) \end{aligned}$$

But

$$\begin{aligned} &\mathsf{E} \mathbf{v}_{\lambda+1} = 0, \quad \mathsf{E} v_- = v_+, \quad \mathsf{EF} = \frac{\mathsf{q}^{\mathsf{h}} - \mathsf{q}^{-\mathsf{h}}}{\mathsf{q} - \mathsf{q}^{-1}} + \mathsf{FE}, \\ &\mathsf{h} \mathbf{v}_{\lambda+1} = (\lambda+1) \mathbf{v}_{\lambda+1} \quad \text{and} \quad \mathsf{h} v_- = -v_-, \end{aligned}$$

so we obtain

$$0 = \mathsf{q}^{-1} \mathbf{v}_{\lambda+1} \otimes v_+ + a(\lambda, \mathsf{q}) \frac{\mathsf{q}^{\lambda+1} - \mathsf{q}^{-(\lambda+1)}}{\mathsf{q} - \mathsf{q}^{-1}} \mathbf{v}_{\lambda+1} \otimes v_+.$$

Hence,

$$\begin{aligned} 0 &= \mathsf{q}^{-1} + a(\lambda, \mathsf{q}) \frac{\mathsf{q}^{\lambda+1} - \mathsf{q}^{-(\lambda+1)}}{\mathsf{q} - \mathsf{q}^{-1}} \\ \Rightarrow a(\lambda, \mathsf{q}) &= \frac{\mathsf{q}^{-1} - \mathsf{q}}{\mathsf{q}\left(\mathsf{q}^{\lambda+1} - \mathsf{q}^{-(\lambda+1)}\right)}. \end{aligned}$$

Therefore, we conclude that

$$J_{VV}(v_+ \otimes v_-) = v_+ \otimes v_- + \frac{\mathsf{q}^{-1} - \mathsf{q}}{\mathsf{q}^{2(\lambda+1)} - 1} v_- \otimes v_+.$$

So we see that we can write

$$[J_{VV}(\lambda)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{\mathsf{q}^{-1}-\mathsf{q}}{\mathsf{q}^{2(\lambda+1)}-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the ordered basis of $V \otimes V$ is

$$(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-).$$

Note that the limit of this $J_{VV}(\lambda)$ as $\mathsf{q} \to 1$ is the $J_{VV}(\lambda)$ for $\mathfrak{g}$. Thus $J_{VV}(\lambda)$ for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ is a deformation of $J_{VV}(\lambda)$ for $\mathfrak{g}$.

5.2 Exchange operator for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let V, W be finite-dimensional representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$. As usual, $\mathcal{R}$ will denote the universal R-matrix.

Definition 5.4 *The exchange operator* $R_{VW}(\lambda) : V \otimes W \to V \otimes W$ *is defined by the formula*

$$R_{VW}(\lambda) = J_{VW}(\lambda)^{-1} \mathcal{R}^{21} J_{WV}^{21}(\lambda).$$

Proposition 5.5 *Suppose* v, w *are homogeneous vectors such that*

$$R_{VW}(\lambda)(v \otimes w) = \sum_i v_i \otimes w_i.$$

Then,

$$P_{WV} \mathcal{R}_{WV} (\Phi^w_{\lambda - \mathsf{wt}\, v} \otimes \mathsf{Id}) \Phi^v_\lambda = \sum_i (\Phi^{v_i}_{\lambda - \mathsf{wt}\, w_i} \otimes \mathsf{Id}) \Phi^{w_i}_\lambda P_{VW}.$$

Example 5.6 Once again, take $\mathfrak{g} = \mathfrak{sl}_2$ and V to be the two-dimensional irreducible representation of $\mathfrak{g}$. We know from Example 5.3 that

$$[J_{VV}(\lambda)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{\mathsf{q}^{-1}-\mathsf{q}}{\mathsf{q}^{2(\lambda+1)}-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and from Example 4.58 that

$$[\mathcal{R}_{VV}] = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{-\frac{1}{2}} & q^{-\frac{1}{2}}(q-q^{-1}) & 0 \\ 0 & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}.$$

Therefore,

$$[R_{VV}(\lambda)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{q^{-1}-q}{q^{2(\lambda+1)}-1} & 0 \\ 0 & \frac{q^{-1}-q}{q^{-(2\lambda+2)}-1} & \frac{\left(q^{2(\lambda+1)}-q^2\right)\left(q^{2(\lambda+1)}-q^{-2}\right)}{\left(q^{2(\lambda+1)}-1\right)^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the limit of this $R_{VV}(\lambda)$ as $q \to 1$ is the $R_{VV}(\lambda)$ for $\mathfrak{g}$. Thus $R_{VV}(\lambda)$ for $\mathfrak{U}_q(\mathfrak{g})$ is a deformation of $R_{VV}(\lambda)$ for $\mathfrak{g}$.

5.3 The ABRR equation for $\mathfrak{U}_q(\mathfrak{g})$

Notation 5.7 *We will denote $\mathcal{R}_0 = \mathcal{R}q^{-\sum_i x_i \otimes x_i}$, where $\{x_i\}$ is an orthonormal basis for $\mathfrak{h}$.*

Theorem 5.8 (ABRR equation for $\mathfrak{U}_q(\mathfrak{g})$**)**

1. *The equation*

$$J(\lambda)(\mathrm{Id} \otimes q^{2\theta(\lambda)}) = \mathcal{R}_0^{21}(\mathrm{Id} \otimes q^{2\theta(\lambda)})J(\lambda)$$

   has a unique solution $J(\lambda)$ of zero weight in $\mathfrak{U}_q(\mathfrak{g})\hat{\otimes}\mathfrak{U}_q(\mathfrak{g})$.
2. *This solution specializes to the fusion operator for $\mathfrak{U}_q(\mathfrak{g})$ on $V \otimes W$ for all finite-dimensional representations V, W of $\mathfrak{U}_q(\mathfrak{g})$.*

We will shortly give a sketch of the proof of this theorem, but first we need to define the quantum analog of the Casimir element.

Definition 5.9 *Let $(\mathbf{H}, R)$ be a quasitriangular Hopf algebra. The Drinfeld element u of $\mathbf{H}$ is defined by the formula*

$$R = \sum_i a_i \otimes b_i \implies u = \sum_i \mathbf{S}(b_i)a_i.$$

Remark 5.10 If $(\mathbf{H}, R) = (\mathfrak{U}_q(\mathfrak{g}), \mathcal{R})$, then u will be in a completion $\widehat{\mathfrak{U}_q(\mathfrak{g})}$ of $\mathfrak{U}_q(\mathfrak{g})$ acting in category $\mathcal{O}$, but not in $\mathfrak{U}_q(\mathfrak{g})$ itself.

Theorem 5.11 (Drinfeld (1990*a*))

1. *u is invertible;*
2. $uxu^{-1} = \mathbf{S}^2(x)$ *for all* $x \in \mathbf{H}$*;*
3. $u^{-1} = \sum_i \mathbf{S}^{-1}(b_i')a_i'$*, where* $\mathcal{R}^{-1} = \sum_i a_i' \otimes b_i'$.

Definition 5.12 *The quantum Casimir element for* $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ *is* $\mathsf{Z} = u\mathsf{q}^{-2\bar{\rho}}$.

Proposition 5.13

1. Z *is central in* $\widehat{\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})}$*;*
2. $\mathsf{Z}|_{M_\lambda} = \mathsf{q}^{-\langle\lambda,\lambda+2\rho\rangle}\,\mathsf{Id}$.

Proof Recall that $\mathbf{S}^2(x) = \mathsf{q}^{2\bar{\rho}}x\mathsf{q}^{-2\bar{\rho}}$ for all $x \in \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$. So, for all $x \in \widehat{\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})}$,

$$x\mathsf{Z} = xu\mathsf{q}^{-2\bar{\rho}} = u\mathbf{S}^{-2}(x)\mathsf{q}^{-2\bar{\rho}} = u\mathsf{q}^{-2\bar{\rho}}x = \mathsf{Z}x,$$

which proves (1). For (2), we note that

$$\mathcal{R} = \mathsf{q}^{\sum_i \mathsf{x}_i \otimes \mathsf{x}_i} + \sum_j a_j \otimes b_j, \quad \text{where } \mathsf{wt}\, a_j > 0,\ \mathsf{wt}\, b_j < 0.$$

Thus

$$u = \mathsf{q}^{-\sum_i \mathsf{x}_i^2} + u', \quad \text{where } u' \in \widehat{\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})} \text{ satisfies } u'\mathbf{v}_\lambda = 0.$$

Hence,

$$\mathsf{Z}\mathbf{v}_\lambda = u\mathsf{q}^{-2\bar{\rho}}\mathbf{v}_\lambda = \mathsf{q}^{-\sum_i \mathsf{x}_i^2 - 2\bar{\rho}}\mathbf{v}_\lambda = \mathsf{q}^{-\langle\lambda,\lambda+2\rho\rangle}\mathbf{v}_\lambda.$$

Since Z is central, it follows that $\mathsf{Z}|_{M_\lambda} = \mathsf{q}^{-\langle\lambda,\lambda+2\rho\rangle}\,\mathsf{Id}$, and (2) is proved. □

We now see why Z is called the quantum Casimir element: it has properties analogous to those of the Casimir element C.

Sketch of proof of Theorem 5.8 In Section 3.5, we proved that the fusion operator for $\mathfrak{g}$ satisfies the ABRR equation by studying the expression

$$F(\lambda) = \left\langle \mathbf{v}^*_{\lambda-\mathsf{wt}\,w-\mathsf{wt}\,v} \otimes \mathsf{Id}, (\Phi^v_{\lambda-\mathsf{wt}\,w} \otimes \mathsf{Id})(\mathsf{C} \otimes \mathsf{Id})\Phi^w_\lambda \mathbf{v}_\lambda \right\rangle, \tag{5.1}$$

where C in the Casimir operator. Perform similar computations with C replaced by Z, the quantum Casimir element, to prove Theorem 5.8. For a complete proof, see Etingof and Schiffmann (2001*a*). □

5.4 Quasi-classical limit for ABRR equation for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$

Recall the ABRR equation for $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$,

$$J(\lambda)(\mathsf{Id} \otimes \mathsf{q}^{2\theta(\lambda)}) = \mathcal{R}_0^{21}(\mathsf{Id} \otimes \mathsf{q}^{2\theta(\lambda)})J(\lambda). \tag{5.2}$$

We would like to study the quasi-classical limit for (5.2). Make the substitutions $\lambda \to \lambda/\hbar, \mathsf{q} \to e^{\varepsilon\hbar/2}$, $\varepsilon \in \mathbf{C}$.

We can write:

$$\tilde{J}(\lambda) = J(\frac{\lambda}{\hbar}) = 1 + \hbar j(\lambda) + O(\hbar^2),$$
$$\mathcal{R} = 1 + \varepsilon\hbar r + O(\hbar^2),$$
$$\mathcal{R}_0 = 1 + \varepsilon\hbar r_0 + O(\hbar^2),$$

where

$$r = \frac{1}{2}\sum_i \mathsf{x}_i \otimes \mathsf{x}_i + \sum_{\alpha\in\mathsf{R}_+} \mathsf{e}_\alpha \otimes \mathsf{e}_{-\alpha}$$

and hence

$$r_0 = r - \frac{1}{2}\sum_i \mathsf{x}_i \otimes \mathsf{x}_i = \sum_{\alpha\in\mathsf{R}_+} \mathsf{e}_\alpha \otimes \mathsf{e}_{-\alpha}.$$

Substituting in (5.2) and extracting the coefficient of $\hbar^1$ gives

$$\begin{aligned} &j(\lambda)(1\otimes e^{\varepsilon\bar{\lambda}}) = (1\otimes e^{\varepsilon\bar{\lambda}}) + \varepsilon r_0^{21}(1\otimes e^{\varepsilon\bar{\lambda}}) \\ \Longrightarrow\ &j(\lambda) - \mathsf{Ad}(1\otimes e^{\varepsilon\bar{\lambda}})j(\lambda) = \varepsilon r_0^{21} = \varepsilon \sum_{\alpha\in\mathsf{R}_+} \mathsf{e}_{-\alpha}\otimes \mathsf{e}_\alpha, \end{aligned} \tag{5.3}$$

where $(\mathsf{Ad}\, x)y = xyx^{-1}$. We write

$$j(\lambda) = \sum_{\beta>0} j^\beta(\lambda), \quad \text{where } j^\beta \in \mathfrak{U}(\mathfrak{g})[-\beta]\otimes\mathfrak{U}(\mathfrak{g})[\beta].$$

We see that if $\mathsf{wt}\,\mathsf{x} = \beta$, then $\mathsf{Ad}\, e^{\varepsilon\bar{\lambda}}\mathsf{x} = e^{\varepsilon\langle\lambda,\beta\rangle}\mathsf{x}$. Thus (5.3) becomes

$$\sum_{\beta>0}\left(1 - e^{\varepsilon\langle\lambda,\beta\rangle}\right) j^\beta(\lambda) = \varepsilon \sum_{\alpha\in\mathsf{R}_+} \mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha.$$

Thus,

$$j^\beta(\lambda) = \begin{cases} \frac{\mathsf{e}_{-\beta}\otimes\mathsf{e}_\beta}{1-e^{\varepsilon\langle\lambda,\beta\rangle}} & \text{if } \beta\in\mathsf{R}_+, \\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$j(\lambda) = \sum_{\alpha\in\mathsf{R}_+} \frac{\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha}{1-e^{\varepsilon\langle\lambda,\alpha\rangle}}. \tag{5.4}$$

Definition 5.14 *Let*

$$\Omega \overset{\text{def}}{=} \sum_i \mathsf{x}_i\otimes\mathsf{x}_i + \sum_{\alpha\in\mathsf{R}_+}(\mathsf{e}_\alpha\otimes\mathsf{e}_{-\alpha} + \mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha).$$

Next, we write $\tilde{R}(\lambda) = 1 - \hbar r(\lambda) + O(\hbar^2)$. Then, from $\tilde{R}(\lambda) = \tilde{J}(\lambda)^{-1}\mathcal{R}^{21}\tilde{J}^{21}(\lambda)$, we obtain

$$\begin{aligned}
r(\lambda) &= j(\lambda) - j^{21}(\lambda) - \varepsilon r^{21} \\
&= \sum_{\alpha\in\mathsf{R}_+} \frac{\varepsilon}{1-e^{\varepsilon\langle\lambda,\alpha\rangle}}\left(\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha - \mathsf{e}_\alpha\otimes\mathsf{e}_{-\alpha}\right) - \varepsilon\sum_i \frac{\mathsf{x}_i\otimes\mathsf{x}_i}{2} + \varepsilon\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha \\
&= -\frac{\varepsilon\Omega}{2} + \sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_\alpha\wedge\mathsf{e}_{-\alpha}\left(\frac{\varepsilon}{2} - \frac{\varepsilon}{1-e^{\varepsilon\langle\lambda,\alpha\rangle}}\right) \\
&= -\frac{\varepsilon\Omega}{2} - \frac{\varepsilon}{2}\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_\alpha\wedge\mathsf{e}_{-\alpha}\frac{1+e^{\varepsilon\langle\lambda,\alpha\rangle}}{1-e^{\varepsilon\langle\lambda,\alpha\rangle}} \\
&= -\frac{\varepsilon\Omega}{2} + \frac{\varepsilon}{2}\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_\alpha\wedge\mathsf{e}_{-\alpha}\,\mathsf{cotanh}\left(\frac{\varepsilon}{2}\langle\lambda,\alpha\rangle\right).
\end{aligned}$$

Definition 5.15 *Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. A classical dynamical r-matrix is $r : \mathfrak{h}^* \to \mathfrak{g}\otimes\mathfrak{g}$ such that*

1. *r has zero weight: $r \in (\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{h}}$.*
2. *r satisfies the classical dynamical Yang–Baxter equation,*

$$\sum_i\left(\frac{\partial r^{12}}{\partial\mathsf{x}_i}\mathsf{x}_i^3 - \frac{\partial r^{13}}{\partial\mathsf{x}_i}\mathsf{x}_i^2 + \frac{\partial r^{23}}{\partial\mathsf{x}_i}\mathsf{x}_i^1\right) + \left[r^{12},r^{13}\right] + \left[r^{12},r^{23}\right] + \left[r^{13},r^{23}\right] = 0.$$

Definition 5.16 *A classical dynamical r-matrix is said to have coupling constant ε if $r(\lambda) + r^{21}(\lambda) = \varepsilon\Omega$.*

Example 5.17 The above computations show that

$$r^{\mathfrak{g}}_{\mathrm{trig}}(\lambda) = -\frac{\varepsilon\Omega}{2} + \frac{\varepsilon}{2}\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_\alpha\wedge\mathsf{e}_{-\alpha}\,\mathsf{cotanh}\left(\frac{\varepsilon}{2}\langle\lambda,\alpha\rangle\right)$$

is a classical dynamical r-matrix; its coupling constant is $-\varepsilon$. It is called the trigonometric solution of (2). Note that

$$r^{\mathfrak{g}}_{\mathrm{rat}}(\lambda) \overset{\mathrm{def}}{=} \lim_{\varepsilon\to 0} r^{\mathfrak{g}}_{\mathrm{trig}}(\lambda) = \sum_{\alpha\in\mathsf{R}_+}\frac{\mathsf{e}_\alpha\wedge\mathsf{e}_{-\alpha}}{\langle\lambda,\alpha\rangle}$$

is also a classical dynamical r-matrix; its coupling constant is zero. It is called the rational solution of (2).

There exist some simple transformations that allow us to obtain new classical dynamical r-matrices from the above.

Definition 5.18 *The gauge transformations are the following:*

1. *Shift of parameters, $r(\lambda) \to r(\lambda - \nu), \nu \in \mathfrak{h}^*$;*
2. *Action of the Weyl group W. Recall that $W = NT/T$, where $T \subset G$ is the maximal torus, and NT is the normalizer of T. So for each $w \in W$, we can choose a lift $\tilde{w} \in NT$; the transformation is $r(\lambda) \to (\tilde{w} \otimes \tilde{w})(r(w^{-1}\lambda))$;*
3. *Let $\phi : \mathfrak{h}^* \to \Lambda^2\mathfrak{h}$ be a function which corresponds to a closed differential 2-form on $\mathfrak{h}^*$. Then $r(\lambda) \to r(\lambda) + \phi(\lambda)$ is a gauge transformation.*

Proposition 5.19 *The set of classical dynamical r-matrices is preserved by gauge transformations.*

We will now see that the *only* classical dynamical r-matrices with coupling constant are essentially those of Example 5.17, up to gauge transformations.

Definition 5.20 *A reductive Lie algebra is a direct sum of a semisimple and an abelian finite dimensional Lie algebra.*

Proposition 5.21 *Suppose $\mathfrak{h} \subset \mathfrak{a} \subset \mathfrak{g}$, where $\mathfrak{g}$ is simple, $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{a}$ a reductive Lie subalgebra. Then the root system of $\mathfrak{a}$ is*

$$\mathsf{R}_\mathfrak{a} = \{\alpha \in \mathsf{R}_\mathfrak{g} | \mathfrak{g}_\alpha \subset \mathfrak{a}\}.$$

Proposition 5.22 *$\mathsf{R}_\mathfrak{a}$ satisfies the following properties:*

1. *$\beta \in \mathsf{R}_\mathfrak{a} \Longrightarrow -\beta \in \mathsf{R}_\mathfrak{a}$;*
2. *$\beta_1, \beta_2 \in \mathsf{R}_\mathfrak{a}, \beta_1 + \beta_2 \in \mathsf{R}_\mathfrak{g} \Longrightarrow \beta_1 + \beta_2 \in \mathsf{R}_\mathfrak{a}$.*

Conversely, any subset satisfying these properties is of the form $\mathsf{R}_\mathfrak{a}$ for a suitable reductive $\mathfrak{a}$.

Definition 5.23

$$r^\mathfrak{a}_{\mathrm{rat}}(\lambda) \stackrel{\mathrm{def}}{=} \sum_{\alpha \in \mathsf{R}_{+\mathfrak{a}}} \frac{\mathsf{e}_\alpha \wedge \mathsf{e}_{-\alpha}}{\langle \lambda, \alpha \rangle}.$$

Theorem 5.24 (Etingof and Varchenko (1998*a*)) *Let $r(\lambda)$ be a classical dynamical r-matrix with zero coupling constant. Then $r(\lambda)$ is gauge equivalent to $r^\mathfrak{a}_{\mathrm{rat}}(\lambda)$, where $\mathfrak{a}$ is a reductive Lie algebra, $\mathfrak{h} \subset \mathfrak{a} \subset \mathfrak{g}$.*

Theorem 5.25 (Etingof and Varchenko (1998*a*)) *Let $r(\lambda)$ be a classical dynamical r-matrix with coupling constant $-\varepsilon$. Then $r(\lambda)$ is gauge equivalent to $r^\mathfrak{g}_{\mathrm{trig}}(\lambda)$ or to its limiting case $\lim_{t\to\infty} r^\mathfrak{g}_{\mathrm{trig}}(\lambda - t\nu), \nu \in \mathfrak{h}^*$.*

Remark 5.26 All limits $\lim_{t\to\infty} r^{\mathfrak{g}}_{\text{trig}}(\lambda - t\nu), \nu \in \mathfrak{h}^*$ look like

$$s_X(\lambda) = -\frac{\varepsilon\Omega}{2} + \sum_{\alpha\in\mathsf{R}_+} \phi_\alpha(\lambda)\mathsf{e}_\alpha \wedge \mathsf{e}_{-\alpha},$$

where $X \subset \Pi$ is some subset, and

$$\phi_\alpha(\lambda) = \begin{cases} \frac{\varepsilon}{2}\,\mathsf{cotanh}\,\frac{\varepsilon\lambda}{2} & \text{if } \alpha \in \mathbf{Z}_+ X, \\ \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

In fact, to get the limit corresponding to $\nu \in \mathfrak{h}^*$, we should choose X so that $\langle \nu, \alpha_i \rangle = 0 \iff \alpha_i \in X$.

Remark 5.27 We cannot obtain all solutions with zero coupling constant by taking a limit as $\varepsilon \to 0$ of solutions with a nonzero coupling constant. This is because there exist $\mathfrak{h} \subset \mathfrak{a} \subset \mathfrak{g}$ where $\mathfrak{a}$ is a reductive but not a Levi subalgebra; that is, $\mathfrak{a}$ does not correspond to a Dynkin subdiagram of $\mathfrak{g}$. An example of this is $\mathfrak{a} = \mathfrak{sl}_2 \times \mathfrak{sl}_2, \mathfrak{g} = \mathfrak{sp}_4$.

6

DYNAMICAL R-MATRICES AND INTEGRABLE SYSTEMS

In this chapter, we will make a connection between dynamical R-matrices and integrable systems. We start with a brief discussion of classical and quantum mechanics and of classical and quantum integrable systems.

6.1 Classical mechanics vs. quantum mechanics

In classical mechanics, we have:

- a phase space M (a manifold);
- observables (functions $f \in C^\infty(M)$);
- a symplectic (or, more generally, Poisson) structure,

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

  which is a biderivation that makes $C^\infty(M)$ into a Lie algebra;
- motion is given by a Hamiltonian $H \in C^\infty(M)$, and the equations of motion are the Hamiltonian equations, $\dot{y_i} = \{y_i, H\}$.

In quantum mechanics, we have:

- a Hilbert space of states, $\mathcal{H}$;
- observables are Hermitian operators on $\mathcal{H}$;
- the analog of the Poisson bracket is the usual bracket of operators, $[A, B] = AB - BA$;
- motion is given by a Hamiltonian $H : \mathcal{H} \to \mathcal{H}$, and the equation of motion is the Schrödinger equation, $\dot{\psi} = -iH\psi$.

Definition 6.1 (see Arnold (1989)) *A classical integrable system is a collection of functionally independent (near each point) functions $I_1, \ldots, I_n$, $n = \frac{1}{2}\dim M$ on M such that $\{I_j, I_k\} = 0$ for all j, k.*

If $I_1, \ldots, I_n$ is a classical integrable system and $H = f(I_1, \ldots, I_n)$, then we can solve the system $\dot{y_i} = \{y_i, H\}$ explicitly. Indeed, $\{I_k, H\} = 0 \Longrightarrow dI_k/dt = 0$, which means that any motion is constrained to the level surfaces of $I_1, \ldots, I_n$. This specifies the motion completely, by the Hamiltonian reduction procedure (namely, as explained in Arnold (1989), each integral allows one to lower the order of the system by 2).

"Definition" 6.2 *A quantum integrable system is an "interesting" collection of commuting linear operators $\{A_1, \ldots, A_n\}$ in a vector space, "desirably" having* *small (e.g. finite-dimensional) common eigenspaces. More generally, it could be an "interesting" commutative subalgebra in a noncommutative algebra.*

If $A_1, \dots, A_n$ is a quantum integrable system and $H = f(A_1, \dots, A_n)$, then the Schrödinger equation decomposes into a direct sum of equations on eigenspaces of $A_1, \dots, A_n$. If these eigenspaces are finite-dimensional, then solving the Schrödinger equation reduces to exponentiating finite matrices. So in a sense the equation is completely solvable, which motivates the terminology "quantum integrable system".

6.2 Transfer matrix construction—A method of construction of quantum integrable systems

Definition 6.3 *Let $\mathbf{H}$ be a Hopf algebra. The Grothendieck ring $\mathcal{G}(\mathsf{Rep}\,\mathbf{H})$ of the category of finite-dimensional representations of $\mathbf{H}$ is a free $\mathbf{Z}$-module whose basis is the set of irreducible finite-dimensional representations of $\mathbf{H}$, with the product*

$$V \cdot W \stackrel{\text{def}}{=} \sum_i m_i U_i,$$

where U_i are the constituents in the composition series of $V \otimes W$. Here m_i are the multiplicities of occurrence of U_i.

Let $\mathbf{H}$ be a Hopf algebra and $K = \mathcal{G}(\mathsf{Rep}\,\mathbf{H})$ be the Grothendieck ring of its category of finite-dimensional representations. Let $R \in \mathbf{H} \otimes \mathbf{H}$. Then we can define a group homomorphism $D : K \to \mathbf{H}$ by $D(V) = (\mathsf{tr}|_V \otimes \mathsf{Id})(R)$ for all irreducible representations V of $\mathbf{H}$.

Proposition 6.4 *If $(\mathbf{\Delta} \otimes \mathsf{Id})R = R^{13}R^{23}$ (e.g. R is a quasi-triangular structure), then D is a ring homomorphism.*

Proof

$$\begin{aligned} D(V)D(W) &= ((\mathsf{tr}|_V \otimes \mathsf{Id})(R))\,((\mathsf{tr}|_W \otimes \mathsf{Id})(R)) \\ &= (\mathsf{tr}|_{V\otimes W} \otimes \mathsf{Id})(R^{13}R^{23}) \\ &= (\mathsf{tr}|_{V\otimes W} \otimes \mathsf{Id})\,((\mathbf{\Delta} \otimes \mathsf{Id})(R)) \\ &= D(V \cdot W). \end{aligned}$$

The last equality follows from the definition of the tensor product of representations of Hopf algebras. □

Corollary 6.5 *If $V \otimes W \cong W \otimes V$ in $\mathsf{Rep}\,\mathbf{H}$ (for example, if $\mathbf{H}$ is quasi-triangular), then $[D(V), D(W)] = 0$. In particular, if $V \otimes W \cong W \otimes V$ for all representations V, W of $\mathbf{H}$, then $\{D(V) : V \in \mathcal{G}(\mathsf{Rep}\,\mathbf{H})\}$ is a system of commuting elements of $\mathbf{H}$.*

Example 6.6 If $\mathbf{H} = \mathfrak{U}_{\mathsf{q}}(\hat{\mathfrak{g}})$ (a *quantum affine algebra*) and U is a finite-dimensional representation of $\mathbf{H}$, then the system of operators $D(V)|_U$ is called the *Gaudin model*.

Example 6.7 If we take $\mathbf{H} = \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ with the usual $\mathcal{R}$, then we do not get any interesting examples. Indeed, since $\mathcal{R} = \mathsf{q}^{\sum_i \mathsf{x}_i \otimes \mathsf{x}_i}(1 + \sum_j a_j \otimes b_j)$, with $\mathsf{wt}\, a_j > 0$ and $\mathsf{wt}\, b_j < 0$, for any irreducible representation V of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$, we have

$$\begin{aligned} D(V) &= (\mathsf{tr}|_V \otimes \mathsf{Id})\left(\mathsf{q}^{\sum_i \mathsf{x}_i \otimes \mathsf{x}_i}(1 + \sum_j a_j \otimes b_j)\right) \\ &= (\mathsf{tr}|_V \otimes \mathsf{Id})\mathsf{q}^{\sum_i \mathsf{x}_i \otimes \mathsf{x}_i} \\ &= \sum_{\nu \in \mathfrak{h}^*} \mathsf{dim}\, V[\nu]\mathsf{q}^{\sum_i \nu(\mathsf{x}_i)\mathsf{x}_i} \\ &= \sum_{\nu \in \mathfrak{h}^*} \mathsf{dim}\, V[\nu]\mathsf{q}^{\overline{\nu}}, \end{aligned}$$

which is just the character of V regarded as an element of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{h})$.

However, if we take $\mathbf{H} = \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ and take R to be the exchange matrix $R(\lambda)$, then we do get interesting results. More precisely, we need to modify the transfer matrix construction, since the exchange matrix satisfies not the usual but rather the dynamical quantum Yang–Baxter equation.

6.3 Dynamical transfer matrix construction

Let $\mathbf{H}$ be a Hopf algebra with a quasi-triangular structure $\mathcal{R}$, and suppose $\mathbf{H}$ contains $\mathfrak{U}(\mathfrak{h})$ as a Hopf subalgebra, where $\mathfrak{h}$ is a commutative finite-dimensional Lie algebra. Suppose $J : \mathfrak{h}^* \to (\mathbf{H} \otimes \mathbf{H})^{\mathfrak{h}}$ is a function. Let $R : \mathfrak{h}^* \to (\mathbf{H} \otimes \mathbf{H})^{\mathfrak{h}}$ be given by $R(\lambda) = J^{-1}(\lambda)\mathcal{R}J^{21}(\lambda)$.

Theorem 6.8 *If $J(\lambda)$ satisfies the dynamical twist equation, then $R(\lambda)$ satisfies the quantum dynamical Yang–Baxter equation.*

Proof We can use the first proof of Theorem 3.15. □

Theorem 6.9 *Let V be a finite-dimensional representation of $\mathbf{H}$ such that $V[0] \neq 0$. For every finite-dimensional representation W of $\mathbf{H}$, consider the linear operator $D_W : \mathsf{Fun}(\mathfrak{h}^*, V[0]) \to \mathsf{Fun}(\mathfrak{h}^*, V[0])$ defined by*

$$D_W = \sum_{\nu \in \mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(R(\lambda))\Big|_{V[0]} \cdot T_{-\nu},$$

where $T_\beta : \mathsf{Fun}(\mathfrak{h}^*, V[0]) \to \mathsf{Fun}(\mathfrak{h}^*, V[0])$ is the translation $(T_\beta f)(\lambda) = f(\lambda + \beta)$. Then, for every pair of finite-dimensional representations U, W of $\mathbf{H}$, $D_{W \otimes U} = D_U D_W$ and $D_U D_W = D_W D_U$.

Remark 6.10 Here, $\mathsf{tr}|_{W[\nu]}\, a$ means $\mathsf{tr}\, (P_{W[\nu]} \circ a)\big|_{W[\nu]}$, where $P_{W[\nu]}$ is projection onto $W[\nu]$.

Lemma 6.11

1. $R^{12}(\lambda - h^3)R^{13}(\lambda) = J^{23}(\lambda)^{-1}R^{1,23}(\lambda)J^{23}(\lambda - h^1)$;
2. $R^{23}(\lambda)R^{13}(\lambda - h^2) = J^{12}(\lambda - h^3)^{-1}R^{12,3}(\lambda)J^{12}(\lambda)$.

Proof We use the fact that $J(\lambda)$ satisfies the dynamical twist equation along with the quasi-triangularity of $\mathcal{R}$:

$$\begin{aligned}R^{12}(\lambda - h^3)R^{13}(\lambda) &= J^{12}(\lambda - h^3)^{-1}\mathcal{R}^{21}J^{21}(\lambda - h^3)J^{13}(\lambda)^{-1}\mathcal{R}^{31}J^{31}(\lambda)\\ &= J^{12}(\lambda - h^3)^{-1}\mathcal{R}^{21}J^{21,3}(\lambda)^{-1}J^{2,13}(\lambda)\mathcal{R}^{31}J^{31}(\lambda)\\ &= J^{12}(\lambda - h^3)^{-1}J^{12,3}(\lambda)^{-1}\mathcal{R}^{21}\mathcal{R}^{31}J^{2,31}(\lambda)J^{31}(\lambda)\\ &= J^{23}(\lambda)^{-1}J^{1,23}(\lambda)^{-1}\mathcal{R}^{21}\mathcal{R}^{31}J^{23,1}(\lambda)J^{23}(\lambda - h^1)\\ &= J^{23}(\lambda)^{-1}J^{1,23}(\lambda)^{-1}(\mathsf{Id} \otimes \boldsymbol{\Delta})\mathcal{R}^{\mathrm{op}}J^{23,1}(\lambda)J^{23}(\lambda - h^1)\\ &= J^{23}(\lambda)^{-1}J^{1,23}(\lambda)^{-1}\mathcal{R}^{23,1}J^{23,1}(\lambda)J^{23}(\lambda - h^1)\\ &= J^{23}(\lambda)^{-1}R^{1,23}(\lambda)J^{23}(\lambda - h^1),\end{aligned}$$

which proves (1). The proof of (2) is similar. □

Proof of Theorem 6.9 We have

$$\begin{aligned}D_{W\otimes U} &= \sum_{\nu\in\mathfrak{h}^*}(\mathsf{tr}|_{W\otimes U[\nu]} \otimes \mathsf{Id})(R^{12,3}(\lambda)) \cdot T_{-\nu}\\ &= \sum_{\nu_1,\nu_2\in\mathfrak{h}^*}(\mathsf{tr}|_{W[\nu_1]\otimes U[\nu_2]} \otimes \mathsf{Id})(R^{12,3}(\lambda)) \cdot T_{-\nu_1}T_{-\nu_2}. \qquad (6.1)\end{aligned}$$

By part (2) of lemma 6.11, we have

$$R^{12,3}(\lambda) = J^{12}(\lambda - h^3)R^{23}(\lambda)R^{13}(\lambda - h^2)J^{12}(\lambda)^{-1};$$

but here, the third component of $R^{12,3}$ acts on $V[0]$, so $h^3 = 0$. So (6.1) becomes

$$\begin{aligned}&D_{W\otimes U}\\ &= \sum_{\nu_1,\nu_2\in\mathfrak{h}^*}(\mathsf{tr}|_{W[\nu_1]\otimes U[\nu_2]} \otimes \mathsf{Id})\left(\mathsf{Ad}\, J^{12}(\lambda)\left(R^{23}(\lambda)R^{13}(\lambda - h^2)\right)\right) \cdot T_{-\nu_1}T_{-\nu_2}\\ &= \sum_{\nu_1,\nu_2\in\mathfrak{h}^*}(\mathsf{tr}|_{W[\nu_1]\otimes U[\nu_2]} \otimes \mathsf{Id})\left(R^{23}(\lambda)R^{13}(\lambda - h^2)\right) \cdot T_{-\nu_1}T_{-\nu_2}\\ &= \sum_{\nu_1,\nu_2\in\mathfrak{h}^*}(\mathsf{tr}|_{U[\nu_2]} \otimes \mathsf{Id})(R(\lambda))(\mathsf{tr}|_{W[\nu_1]} \otimes \mathsf{Id})(R(\lambda - \nu_2)) \cdot T_{-\nu_1}T_{-\nu_2}\\ &= \sum_{\nu_1,\nu_2\in\mathfrak{h}^*}(\mathsf{tr}|_{U[\nu_2]} \otimes \mathsf{Id})(R(\lambda)) \cdot T_{-\nu_2}(\mathsf{tr}|_{W[\nu_1]} \otimes \mathsf{Id})(R(\lambda)) \cdot T_{-\nu_1}\\ &= D_U D_W.\end{aligned}$$

Since $W \otimes U \cong U \otimes W$, we see that

$$D_U D_W = D_{W\otimes U} = D_{U\otimes W} = D_W D_U.$$

□

Proposition 6.12 *Let* $\mathbf{H} = \mathfrak{U}_\mathsf{q}(\mathfrak{g})$, *and let* $J(\lambda)$ *and* $R(\lambda)$ *be the fusion and exchange matrices. Let* $V_1, \ldots, V_r$ *be fundamental representations of* $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$ $(r = \mathsf{rank}\,\mathfrak{g})$. *Then:*

1. $D_{V_1}, \ldots, D_{V_r}$ *are algebraically independent;*
2. *for all* $W \in \mathsf{Rep}\,\mathfrak{U}_\mathsf{q}(\mathfrak{g})$, *we have* $D_W \in \mathbf{C}[D_{V_1}, \ldots, D_{V_r}]$.

Before proving the proposition, we need a lemma. We say that $\lambda \to \infty$ if $|\langle \lambda, \alpha_i\rangle| \to \infty$ for all i.

Lemma 6.13 *For* $\mathfrak{U}(\mathfrak{g})$, *we have* $\lim_{\lambda\to\infty} J(\lambda) = \mathsf{Id}$ *and* $\lim_{\lambda\to\infty} R(\lambda) = \mathsf{Id}$.

Proof Write $J(\lambda) = 1 + \sum_{\beta>0}\sum_{j=1}^{m_\beta} \phi_j^\beta \otimes \psi_j^\beta$, where $\mathsf{wt}\,\phi_j^\beta = -\beta$ and $\mathsf{wt}\,\psi_j^\beta = \beta$. Since $J(\lambda)$ satisfies the ABRR equation, we know that

$$\mathsf{ad}(\mathsf{Id} \otimes \theta(\lambda))J(\lambda) = -\left(\sum_{\alpha\in\mathsf{R}_+} \mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha\right)(\mathsf{Id} + J(\lambda)).$$

For $\gamma > 0$, we can extract the terms of the form $A \otimes B$ with $\mathsf{wt}\,A = -\gamma$ and $\mathsf{wt}\,B = \gamma$. We see that

$$\mathsf{ad}(\mathsf{Id} \otimes \theta(\lambda))\left(\sum_{j=1}^{m_\beta} \phi_j^\gamma \otimes \psi_j^\gamma\right) = -\sum_{\gamma\geq\alpha>0} \mathsf{e}_{-\alpha}\phi_j^{\gamma-\alpha} \otimes \mathsf{e}_\alpha\psi_j^{\gamma-\alpha},$$

where $m_0 = 1, \phi_1^0 = \psi_1^0 = 1$. Using this recursively, we see that

$$\lim_{\lambda\to\infty}\sum_{j=1}^{m_\beta} \phi_j^\beta \otimes \psi_j^\beta = 0$$

for all $\beta > 0$. Thus $\lim_{\lambda\to\infty} J(\lambda) = \mathsf{Id}$. Clearly, this implies $\lim_{\lambda\to\infty} R(\lambda) = \mathsf{Id}$.

□

Proof of Proposition 6.12 It is enough to prove (1) for the case $\mathsf{q} = 1$. For any $W \in \mathsf{Rep}\,\mathfrak{U}(\mathfrak{g})$, write $D_W^{(\infty)} \stackrel{\text{def}}{=} \lim_{\mu\to\infty} T_{-\mu} D_W T_\mu$; then, it is enough to show that $D_{V_1}^{(\infty)}, \ldots, D_{V_r}^{(\infty)}$ are algebraically independent. For all $W \in \mathsf{Rep}\,\mathfrak{U}(\mathfrak{g})$, we have

$$\begin{aligned} D_W^{(\infty)} &= \lim_{\mu\to\infty} T_{-\mu} D_W T_\mu \\ &= \lim_{\mu\to\infty} \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(R(\lambda-\mu)) \cdot T_{-\nu} \\ &= \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id}) T_{-\nu} \quad \text{by Lemma 6.13} \\ &= \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu] T_{-\nu}, \end{aligned}$$

and the algebraic independence of $D_{V_1}^{(\infty)}, \ldots, D_{V_r}^{(\infty)}$ now follows from the known fact that the characters $\chi_{V_1}, \ldots, \chi_{V_r}$ of the fundamental representations are algebraically independent. So (1) is proved. (2) follows from Theorem 6.9. □

Remark 6.14 This means that $D_{V_1}, \ldots, D_{V_r}$ form a "quantum integrable system".

Proposition 6.15 *Suppose* $0 < |\mathsf{q}| < 1$*; then*

1. $\lim_{\lambda\to+\infty} J(\lambda) = \mathcal{R}_0^{21}$;
2. $\lim_{\lambda\to-\infty} J(\lambda) = \mathsf{Id}$.

Here, the limit as "$\lambda \to +\infty$*" is taken over* λ *in the dominant chamber (*$\mathsf{Re}\, \langle\lambda, \alpha_i\rangle \to +\infty$ *for all* i*), and similarly for the limit as "*$\lambda \to -\infty$*".*

Proof We write

$$J(\lambda) = \sum_{\beta\geq 0} J^{(\beta)}(\lambda), (\mathcal{R}_0^{21})^{-1} = \sum_{\beta\geq 0} S^{(\beta)},$$

where $J^{(\beta)}, S^{(\beta)} \in \mathfrak{U}_\mathsf{q}(\mathfrak{g})[-\beta] \otimes \mathfrak{U}_\mathsf{q}(\mathfrak{g})[\beta]$, $J^{(0)}(\lambda) = S^{(0)}(\lambda) = 1$. It can be shown that the limits exist (by considering recursive relations for $J^{(\beta)}$). Let us compute them. For (1):

$$\begin{aligned} &\lim_{\lambda\to+\infty} J(\lambda) \\ &= \lim_{\lambda\to+\infty} \mathcal{R}_0^{21} (\mathsf{Id} \otimes \mathsf{q}^{2\theta(\lambda)}) J(\lambda) (\mathsf{Id} \otimes \mathsf{q}^{-2\theta(\lambda)}) \quad \text{(from ABRR)} \\ &= \mathcal{R}_0^{21} \lim_{\lambda\to+\infty} \sum_{\beta\geq 0} \mathsf{Ad}(\mathsf{Id} \otimes \mathsf{q}^{2(\bar\lambda+\bar\rho)}) \Big((\mathsf{Id} \otimes \mathsf{q}^{-\sum_i \mathsf{x}_i^2}) J^{(\beta)}(\lambda) (\mathsf{Id} \otimes \mathsf{q}^{\sum_i \mathsf{x}_i^2}) \Big) \\ &= \mathcal{R}_0^{21} \lim_{\lambda\to+\infty} \sum_{\beta\geq 0} \mathsf{q}^{2\langle\lambda+\rho,\beta\rangle} \Big((\mathsf{Id} \otimes \mathsf{q}^{-\sum_i \mathsf{x}_i^2}) J^{(\beta)}(\lambda) (\mathsf{Id} \otimes \mathsf{q}^{\sum_i \mathsf{x}_i^2}) \Big) \\ &= \mathcal{R}_0^{21} \quad \text{(since all terms tend to 0 except for the } \beta = 0 \text{ term).} \end{aligned}$$

For (2):

$$\begin{aligned}
&\lim_{\lambda\to-\infty} J(\lambda)\\
&= \lim_{\lambda\to-\infty} (\mathsf{Id}\otimes\mathsf{q}^{-2\theta(\lambda)})(\mathcal{R}_0^{21})^{-1}J(\lambda)(\mathsf{Id}\otimes\mathsf{q}^{2\theta(\lambda)}) \quad \text{(from ABRR)}\\
&= \lim_{\lambda\to-\infty} \sum_{\beta,\gamma\geq 0} \mathsf{Ad}(\mathsf{Id}\otimes\mathsf{q}^{-2\bar\lambda-2\bar\rho}) \left((\mathsf{Id}\otimes\mathsf{q}^{\sum_i \mathsf{x}_i^2})S^\gamma(\lambda)J^\beta(\lambda)(\mathsf{Id}\otimes\mathsf{q}^{-\sum_i \mathsf{x}_i^2})\right)\\
&= \lim_{\lambda\to-\infty} \sum_{\beta,\gamma\geq 0} \mathsf{q}^{2\langle\lambda+\rho,\beta+\gamma\rangle} \left((\mathsf{Id}\otimes\mathsf{q}^{\sum_i \mathsf{x}_i^2})S^\gamma(\lambda)J^\beta(\lambda)(\mathsf{Id}\otimes\mathsf{q}^{-\sum_i \mathsf{x}_i^2})\right)\\
&= 1 \quad \text{(since all terms tend to 0 except for the } \beta=\gamma=0 \text{ term).}
\end{aligned}$$

□

Corollary 6.16 $\lim_{\lambda\to-\infty} R(\lambda) = \mathcal{R}^{21}$ *and* $\lim_{\lambda\to+\infty} R(\lambda) = \mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\mathcal{R}_0$.

Proof This follows from Proposition 6.15 and from the equation

$$R(\lambda) = J(\lambda)^{-1}\mathcal{R}^{21}J^{21}(\lambda).$$

□

Proposition 6.17 *For every finite-dimensional representation W of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$, we have:*
$\lim_{\lambda\to-\infty} D_W = \lim_{\lambda\to+\infty} D_W = \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]T_{-\nu}$;

Proof

$$\begin{aligned}
&\lim_{\lambda\to-\infty} D_W\\
&= \lim_{\lambda\to-\infty} \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]}\otimes\mathsf{Id})(R(\lambda))\cdot T_{-\nu}\\
&= \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]}\otimes\mathsf{Id})\mathcal{R}^{21}\cdot T_{-\nu} \quad \text{by corollary 6.16}\\
&= \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]}\otimes\mathsf{Id})\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\cdot T_{-\nu} \quad \text{since } (\mathsf{tr}|_{W[\nu]}\otimes\mathsf{Id})\mathcal{R}^{21} \text{ acts on weight 0}\\
&= \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]}\otimes\mathsf{Id})(\mathsf{Id}\otimes\mathsf{Id})\cdot T_{-\nu} \quad \text{since } \mathsf{x}_i \text{ acts as 0 on} V[0]\\
&= \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]T_{-\nu}.
\end{aligned}$$

The proof for $\lim_{\lambda\to+\infty} D_W$ is similar. □

Corollary 6.18

1. *The coefficients of* D_W *are rational functions of* $x_i = \mathsf{q}^{\langle\lambda,\alpha_i\rangle}$, *and are regular at* $x_i = 0, i = 1, \ldots, r$. *Hence they can be viewed as elements of* $\mathbf{C}[[\mathsf{q}^{\langle\lambda,\alpha_1\rangle}, \ldots, \mathsf{q}^{\langle\lambda,\alpha_r\rangle}]]$.
2. *The coefficients of* D_W *are rational functions of* $y_i = \mathsf{q}^{-\langle\lambda,\alpha_i\rangle}$, *and are regular at* $y_i = 0, i = 1, \ldots, r$. *Hence they can be viewed as elements of* $\mathbf{C}[[\mathsf{q}^{-\langle\lambda,\alpha_1\rangle}, \ldots, \mathsf{q}^{-\langle\lambda,\alpha_r\rangle}]]$.

7

TRACES OF INTERTWINERS FOR $\mathfrak{U}_q(\mathfrak{g})$

7.1 Generalized Macdonald–Ruijsenaars operators

Definition 7.1 *Given a finite-dimensional representation W of $\mathfrak{U}_q(\mathfrak{g})$, we will write $\mathbf{R}(\lambda) = R(-\lambda - \rho)$ and we will define the generalized Macdonald–Ruijsenaars operator $\mathbf{D}_W : \mathsf{Fun}(\mathfrak{h}^*, V[0]) \to \mathsf{Fun}(\mathfrak{h}^*, V[0])$ by the formula*

$$\mathbf{D}_W = \sum_{\nu \in \mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(\mathbf{R}(\lambda)) \cdot T_\nu.$$

Let $Y_\mu = \mathsf{q}^{-2\langle \lambda, \mu \rangle} \mathbf{C}[[\mathsf{q}^{-\langle \lambda, \alpha_i \rangle}, i = 1, \ldots, r]] \otimes V[0]$. By Corollary 6.18, the coefficients of $\mathbf{D}_W$ have a Taylor expansion in $\mathsf{q}^{-\langle \lambda, \alpha_i \rangle}$; hence, $\mathbf{D}_W$ acts as an operator $\mathbf{D}_W : Y_\mu \to Y_\mu$ which preserves the subspaces $Y_{\mu+\beta} \subset Y_\mu, \beta \in \mathsf{Q}_+$. In particular, we get an operator $\mathbf{D}$ on $Y_\mu / \sum_i Y_{\mu+\alpha_i} \cong \mathsf{q}^{-2\langle \lambda, \mu \rangle} V[0] \cong V[0]$.

Proposition 7.2 $\mathbf{D}$ *acts on $Y_\mu / \sum_i Y_{\mu+\alpha_i}$ as $\chi_W(\mathsf{q}^{-2\bar{\mu}})\,\mathsf{Id}$.*

Proof For $\mathsf{q}^{-2\langle \lambda, \mu \rangle} v \in Y_\mu / \sum_i Y_{\mu+\alpha_i}$, we have

$$\begin{aligned}
&\mathbf{D}_W \mathsf{q}^{-2\langle \lambda, \mu \rangle} v \\
&\quad = \mathbf{D}_W|_{\mathsf{q}^{-\langle \lambda, \alpha_i \rangle}=0}\, \mathsf{q}^{-2\langle \lambda, \mu \rangle} v \\
&\quad = \left(\lim_{\lambda \to +\infty} \mathbf{D}_W \right) \mathsf{q}^{-2\langle \lambda, \mu \rangle} v \\
&\quad = \left(\sum_{\nu \in \mathfrak{h}^*} \mathsf{dim}\, W[\nu] T_\nu \right) \mathsf{q}^{-2\langle \lambda, \mu \rangle} v \quad \text{(argument similar to Proposition 6.17)} \\
&\quad = \left(\sum_{\nu \in \mathfrak{h}^*} \mathsf{dim}\, W[\nu] \mathsf{q}^{-2\langle \nu, \mu \rangle} \right) \mathsf{q}^{-2\langle \lambda, \mu \rangle} v \\
&\quad = \chi_W(\mathsf{q}^{-2\bar{\mu}}) \mathsf{q}^{-2\langle \lambda, \mu \rangle} v.
\end{aligned}$$

□

Lemma 7.3 *For generic μ, the r-tuple $\left(\chi_{V_1}(\mathsf{q}^{-2(\overline{\mu+\beta})}), \ldots, \chi_{V_r}(\mathsf{q}^{-2(\overline{\mu+\beta})})\right)$ determines $\beta \in \mathsf{Q}$ uniquely.*

Proof Such an r-tuple uniquely determines $\mathsf{q}^{-2(\overline{\mu+\beta})} \in T/\mathbf{W}$, but for generic μ the map $\mathsf{Q} \to T/\mathbf{W}$ given by $\beta \mapsto \mathsf{q}^{-2(\overline{\mu+\beta})}$ is injective. (*Note:* Here $\mathbf{W}$ denotes the Weyl group of $\mathfrak{g}$ and T denotes the maximal torus of G.) □

Lemma 7.4 *Let U be a finite-dimensional vector space over $\mathbf{C}$, and let $A_i : U \to U, i \in I$ be commuting operators (here I is an index set). Let*

$$U = F_0U \supset F_1U \supset F_2U \supset \cdots \supset F_mU = 0$$

be a filtration preserved by each A_i. Suppose that $A_i|_{F_jU/F_{j+1}U} = \lambda_j(i)\,\mathsf{Id}, j = 0, \ldots, m-1$, and the functions $\lambda_j : I \to \mathbf{C}$ are distinct. Then there exists a basis B of U such that A_i is diagonal with respect to B for all $i \in I$ and B is consistent with the filtration (i.e. $B \cap F_jU$ is a basis of F_jU for all $j = 0, \ldots, m$).

Proof The proof is by induction on m, the length of the filtration. For $m = 1$ the result is trivial. Suppose the result is true for $m < m_0$. Then there is a basis B' of F_1U such that $A_i|_{F_1U}$ is diagonal with respect to B' for all $i \in I$ and B' is consistent with the filtration $F_1U \supset F_2U \supset \cdots \supset F_mU = 0$. Complete B' to a basis B of U, and let $B_0 = B \setminus B'$.

Let $B_1 = B \cap F_1U \setminus B \cap F_2U$. Since the functions λ_j are distinct, there must exist $i_0 \in I$ such that $\lambda_0(i_0) \neq \lambda_1(i_0)$. Subtracting linear combinations of the elements of B_1 from the elements of B_0, we obtain a new basis $B = B' \cup B_0$ of U such that $A_{i_0}(B_0) \subset \mathsf{span}\, B_0 \oplus F_2U$. Since the A_i commute, we see that $A_i(B_0) \subset \mathsf{span}\, B_0 \oplus F_2U$ for all $i \in I$. Continuing in this way, we obtain the desired basis B. □

Theorem 7.5 *For generic μ, the operators $\mathbf{D}_W$ are simultaneously diagonalizable on Y_μ. Furthermore, there exists a unique $\hat{F}_V(\lambda, \mu) \in Y_\mu \otimes V[0]^* = \mathsf{q}^{-2\langle\lambda,\mu\rangle}\mathbf{C}[[\mathsf{q}^{-\langle\lambda,\alpha_1\rangle}, \ldots, \mathsf{q}^{-\langle\lambda,\alpha_r\rangle}]] \otimes \mathsf{End}\, V[0]$ such that*

$$\hat{F}(\lambda, \mu) = \mathsf{q}^{-2\langle\lambda,\mu\rangle}\, \mathsf{Id}_{V[0]} + \text{terms of lower order}$$

and

$$\mathbf{D}_W^{(\lambda)} \hat{F}(\lambda, \mu) = \chi_W(\mathsf{q}^{-2\bar{\mu}})\hat{F}(\lambda, \mu).$$

Proof This is simply a consequence of Proposition 7.2 and the infinite-dimensional versions of Lemmas 7.3 and 7.4. □

Remark 7.6 The eigenvectors of $\mathbf{D}_W$ in Y_μ are then $\hat{F}_V(\lambda, \mu + \beta)v, \beta \in \mathsf{Q}_+, v \in V[0]$.

7.2 Construction of $F_V(\lambda, \mu)$

For all $v \in V[0]$, let $\Phi_\mu^v : M_\mu \to M_\mu \otimes V$ be such that $\langle \Phi_\mu^v \rangle = v$. Let

$$\Psi^v(\lambda, \mu) = (\mathsf{tr}|_{M_\mu} \otimes \mathsf{Id})(\Phi_\mu^v \mathsf{q}^{2\bar{\lambda}}) \in V[0] \otimes \mathsf{q}^{2\langle \lambda, \mu \rangle} \mathbf{C}[[\mathsf{q}^{-2\langle \lambda, \alpha_1 \rangle}, \ldots, \mathsf{q}^{-2\langle \lambda, \alpha_r \rangle}]].$$

Let $\{v_i\}$ be a basis of $V[0]$, and let

$$\Psi_V(\lambda, \mu) = \sum_i \Psi^{v_i}(\lambda, \mu) \otimes v_i^* \in \mathsf{End}\, V[0] \otimes \mathsf{q}^{2\langle \lambda, \mu \rangle} \mathbf{C}[[\mathsf{q}^{-2\langle \lambda, \alpha_1 \rangle}, \ldots, \mathsf{q}^{-2\langle \lambda, \alpha_r \rangle}]].$$

Let $\delta_{\mathsf{q}}(\lambda) = \mathsf{q}^{2\langle \lambda, \rho \rangle} \prod_{\alpha \in \mathsf{R}_+} \left(1 - \mathsf{q}^{-2\langle \lambda, \alpha \rangle}\right)$. Let

$$\mathbf{J}(\lambda) = J(-\lambda - \rho),$$

and define $\mathbf{Q}(\lambda) : V \to V$ as follows:

$$\text{if } \mathbf{J}(\lambda) = \sum_i a_i \otimes b_i, \quad \text{then } \mathbf{Q}(\lambda) = \sum_i \mathbf{S}^{-1}(b_i) a_i.$$

We then write

$$F_V(\lambda, \mu) \stackrel{\text{def}}{=} \Psi_V(\lambda, -\mu - \rho) \delta_{\mathsf{q}}(\lambda) \mathbf{Q}^{-1}(\mu).$$

In the next few sections, we will prove the following results about the *trace function* $F_V(\lambda, \mu)$.

Theorem 7.7 (Etingof and Varchenko (2000))

1. *$F_V(\lambda, \mu)$ satisfies the Macdonald–Ruijsenaars equations; i.e.,*

$$\mathbf{D}_W^{(\lambda)} F_V(\lambda, \mu) = \chi_W(\mathsf{q}^{-2\bar{\mu}}) F_V(\lambda, \mu).$$

Hence $\hat{F} = F$.

2. *$F_V(\lambda, \mu)$ satisfies the dual Macdonald–Ruijsenaars equations; i.e.,*

$$\mathbf{D}_W^{(\mu)} F_V(\lambda, \mu) = \chi_W(\mathsf{q}^{-2\bar{\lambda}}) F_V(\lambda, \mu),$$

where $\mathbf{D}_W^{(\mu)}$ acts in the $V^*[0]$ component.

3. *$F_V(\lambda, \mu)$ has the symmetry property; i.e., $F_V(\lambda, \mu) = F_{V^*}^*(\mu, \lambda)$.*

7.3 Quantum spin Calogero–Moser Hamiltonian

Let $\mathsf{q} = e^{\hbar/2}$. Let S denote the substitution $\lambda \to \lambda/\hbar$. Let $\tilde{\mathbf{D}}_W \stackrel{\text{def}}{=} S \circ \mathbf{D}_W \circ S^{-1}$.

Proposition 7.8 $\tilde{\mathbf{D}}_W = \sum_{\nu \in \mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(\mathbf{R}(\frac{\lambda}{\hbar})) \cdot T_{\hbar\nu}$.

Proof For $g : \mathfrak{h}^* \to V[0]$, we have:

$$\begin{aligned}
\tilde{\mathbf{D}}_W g &= \sum_{\nu \in \mathfrak{h}^*} \left(S \circ (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(\mathbf{R}(\lambda)) T_\nu \circ S^{-1} \right) g(\lambda) \\
&= \sum_{\nu \in \mathfrak{h}^*} \left(S \circ (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(\mathbf{R}(\lambda)) T_\nu \right) g(\hbar\lambda) \\
&= \sum_{\nu \in \mathfrak{h}^*} \left(S \circ (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})(\mathbf{R}(\lambda)) \right) g(\hbar(\lambda + \nu)) \\
&= \sum_{\nu \in \mathfrak{h}^*} \left((\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id}) \left(\mathbf{R}(\frac{\lambda}{\hbar}) \right) \right) g(\lambda + \hbar\nu) \\
&= \sum_{\nu \in \mathfrak{h}^*} \left((\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id}) \left(\mathbf{R}(\frac{\lambda}{\hbar}) \right) \cdot T_{\hbar\nu} \right) g(\lambda),
\end{aligned}$$

and the result follows. □

Corollary 7.9 $\lim_{\hbar \to 0} \tilde{\mathbf{D}}_W = (\dim W)\, \mathsf{Id}$.

Proposition 7.10

$$\begin{aligned}
\tilde{\mathbf{D}}_W &= (\dim W)\, \mathsf{Id} \\
&+ \hbar^2 \left(\sum_{\nu \in \mathfrak{h}^*} \dim W[\nu] \frac{\partial_\nu^2}{2} - \sum_{\alpha \in \mathsf{R}_+} (\mathsf{tr}|_W \, \mathsf{e}_{-\alpha} \mathsf{e}_\alpha) \cdot \frac{\mathsf{e}_\alpha \mathsf{e}_{-\alpha}}{\left(e^{\langle \alpha, \lambda \rangle/2} - e^{-\langle \alpha, \lambda \rangle/2} \right)^2} \right) \\
&+ O(\hbar^3),
\end{aligned}$$

where ∂_ν denotes the directional derivative with respect to ν.

Proof Recall (from Section 5.4) the Taylor expansions of $J(\lambda/\hbar)$ and $\mathcal{R}$:

$$J\left(\frac{\lambda}{\hbar}\right) = 1 + \hbar j(\lambda) + O(\hbar^2), \quad \text{where } j(\lambda) = \sum_{\alpha \in \mathsf{R}_+} \frac{\mathsf{e}_{-\alpha} \otimes \mathsf{e}_\alpha}{1 - e^{\langle \lambda, \alpha \rangle}};$$

$$\mathcal{R} = 1 + \hbar r + \hbar^2 r_2 + O(\hbar^3), \quad \text{where } r = \frac{1}{2} \sum_i \mathsf{x}_i \otimes \mathsf{x}_i + \sum_{\alpha \in \mathsf{R}_+} \mathsf{e}_\alpha \otimes \mathsf{e}_{-\alpha}.$$

We have

$$\begin{aligned}
\mathbf{J}\left(\frac{\lambda}{\hbar}\right) &= J\left(-\frac{\lambda}{\hbar} - \rho\right) \\
&= J\left(\frac{\lambda - \hbar\rho}{\hbar}\right) \\
&= 1 + \hbar j(-\lambda - \hbar\rho) + O(\hbar^2) \\
&= 1 + \hbar j(-\lambda) + \hbar^2 j_2 + O(\hbar^3) \quad \text{for some } j_2;
\end{aligned}$$

thus,

$$\mathbf{J}^{-1}\left(\frac{\lambda}{\hbar}\right) = 1 - \hbar j(-\lambda) + \hbar^2 \tilde{j}_2 + O(\hbar^3) \quad \text{for some } \tilde{j}_2.$$

Hence,

$$\begin{aligned}\mathbf{R}\left(\frac{\lambda}{\hbar}\right) &= \mathbf{J}^{-1}\left(\frac{\lambda}{\hbar}\right)\mathcal{R}^{21}\mathbf{J}^{21}\left(\frac{\lambda}{\hbar}\right)\\ &= (1 - \hbar j(-\lambda) + \hbar^2\tilde{j}_2 + O(\hbar^3))\\ &\quad \times(1 + \hbar r^{21} + \hbar^2 r_2^{21} + O(\hbar^3))(1 + \hbar j^{21}(-\lambda) + \hbar^2 j_2^{21} + O(\hbar^3)).\end{aligned}$$

Now write $(\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\mathbf{R}\left(\frac{\lambda}{\hbar}\right) = \mathfrak{t}_0^{(\nu)} + \mathfrak{t}_1^{(\nu)}\hbar + \mathfrak{t}_2^{(\nu)}\hbar^2 + O(\hbar^3)$. It is clear that $\mathfrak{t}_0^{(\nu)} = \dim W[\nu]$. We see that:

$$\begin{aligned}\mathfrak{t}_1^{(\nu)} &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left(-j(-\lambda) + r^{21} + j^{21}(-\lambda)\right)\\ &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})r^{21} \quad \text{since } j \text{ is strictly lower triangular}\\ &= \frac{1}{2}(\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\sum_i \mathsf{x}_i \otimes \mathsf{x}_i\\ &= 0 \quad \text{since } \mathsf{x}_i \text{ acts as 0 on } V[0]; \text{ and furthermore,}\end{aligned}$$

$$\begin{aligned}\mathfrak{t}_2^{(\nu)} &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left(\tilde{j}_2 + r_2^{21} + j_2^{21} - j(-\lambda)r^{21} - j(-\lambda)j^{21}(-\lambda) + r^{21}j^{21}(-\lambda)\right)\\ &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left(-j(-\lambda)r^{21} - j(-\lambda)j^{21}(-\lambda) + r^{21}j^{21}(-\lambda)\right)\\ &\qquad \text{(the argument is similar to the one that shows } \mathfrak{t}_1^{(\nu)} = 0)\\ &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left((-j(-\lambda) + r^{21})j^{21}(-\lambda)\right)\\ &\qquad \text{(since } j(-\lambda)r^{21} \text{ is strictly lower triangular)}\\ &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left(\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha\left(1 - \frac{1}{1 - e^{-\langle\lambda,\alpha\rangle}}\right) + \frac{1}{2}\sum_i \mathsf{x}_i\otimes\mathsf{x}_i\right)\\ &\quad \times\left(\sum_{\alpha'\in\mathsf{R}_+}\frac{\mathsf{e}_{\alpha'}\otimes\mathsf{e}_{-\alpha'}}{1 - e^{-\langle\lambda,\alpha'\rangle}}\right)\\ &= (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left(\sum_{\alpha\in\mathsf{R}_+}\mathsf{e}_{-\alpha}\otimes\mathsf{e}_\alpha\left(1 - \frac{1}{1 - e^{-\langle\lambda,\alpha\rangle}}\right)\frac{\mathsf{e}_\alpha\otimes\mathsf{e}_{-\alpha}}{1 - e^{-\langle\lambda,\alpha\rangle}}\right)\\ &\qquad \text{(since } \mathsf{wt}\, x \neq 0 \Longrightarrow \mathsf{tr}|_{W[\nu]}\, x = 0)\\ &= -\sum_{\alpha\in\mathsf{R}_+}\left(\mathsf{tr}|_{W[\nu]}\,\mathsf{e}_{-\alpha}\mathsf{e}_\alpha\right)\cdot\frac{\mathsf{e}_\alpha\mathsf{e}_{-\alpha}}{\left(e^{\langle\alpha,\lambda\rangle/2} - e^{-\langle\alpha,\lambda\rangle/2}\right)^2}.\end{aligned}$$

Then $T_{\hbar\nu} = e^{\hbar\partial_\nu} = 1 + \hbar\partial_\nu + \hbar^2\partial_\nu^2/2 + O(\hbar^3)$ by Taylor's formula. Now write $\tilde{\mathbf{D}}_W = \mathfrak{d}_0 + \hbar\mathfrak{d}_1 + \hbar^2\mathfrak{d}_2 + O(\hbar^3)$. We see that

$$\begin{aligned}\tilde{\mathbf{D}}_W &= \sum_{\nu\in\mathfrak{h}^*} (\mathsf{tr}|_{W[\nu]} \otimes \mathsf{Id})\left(\mathbf{R}\left(\frac{\lambda}{\hbar}\right)\right)\cdot T_{\hbar\nu} \quad \text{by Proposition 7.8}\\ &= \sum_{\nu\in\mathfrak{h}^*}\left(\mathsf{dim}\, W[\nu] + \hbar^2 \mathfrak{t}_2^{(\nu)} + O(\hbar^3)\right)\left(1 + \hbar\partial_\nu + \frac{\hbar^2\partial_\nu^2}{2} + O(\hbar^3)\right).\end{aligned}$$

Thus $\mathfrak{d}_0 = \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu] = \mathsf{dim}\, W$ and $\mathfrak{d}_1 = \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\partial_\nu = \partial_\beta$, where $\beta = \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\nu \in \mathfrak{h}^*$. But β is invariant under the action of the Weyl group, and $(\mathfrak{h}^*)^{\mathbf{W}} = 0$, so $\beta = 0$. Therefore, $\mathfrak{d}_1 = 0$. Finally, we see that

$$\begin{aligned}\mathfrak{d}_2 &= \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\frac{\partial_\nu^2}{2} + \mathfrak{t}_2^{(\nu)}\\ &= \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\frac{\partial_\nu^2}{2} - \sum_{\alpha\in\mathsf{R}_+} (\mathsf{tr}|_W\, \mathsf{e}_{-\alpha}\mathsf{e}_\alpha)\cdot \frac{\mathsf{e}_\alpha \mathsf{e}_{-\alpha}}{\left(e^{\langle\alpha,\lambda\rangle/2} - e^{-\langle\alpha,\lambda\rangle/2}\right)^2},\end{aligned}$$

and the result follows. □

Now, for all $W \in \mathsf{Rep}\,\mathfrak{g}$, we can define a form B_W on G by $B_W(\mathsf{x},\mathsf{y}) \stackrel{\text{def}}{=} \mathsf{tr}|_W\, \mathsf{xy}$. We see that $B_W(\mathsf{x},\mathsf{y}) = \gamma_W \langle \mathsf{x},\mathsf{y}\rangle$, where γ_W is a constant.

Lemma 7.11 *Let $\{\mathsf{x}_i\}$ be an orthonormal basis of $\mathfrak{h}$. Then*

$$\sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\nu\otimes\nu = \gamma_W \sum_i \mathsf{x}_i^*\otimes \mathsf{x}_i^*,$$

where $\{\mathsf{x}_i^*\}$ is the basis of $\mathfrak{h}^*$ dual to $\{\mathsf{x}_i\}$.

Proof For every $\mathsf{a},\mathsf{b}\in\mathfrak{h}$, we have

$$\begin{aligned}\left\langle \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\nu\otimes\nu, \mathsf{a}\otimes\mathsf{b}\right\rangle &= \sum_{\nu\in\mathfrak{h}^*} \mathsf{dim}\, W[\nu]\nu(\mathsf{a})\nu(\mathsf{b})\\ &= \mathsf{tr}|_W\, \mathsf{ab}\\ &= \gamma_W \langle \mathsf{a},\mathsf{b}\rangle\\ &= \gamma_W \left\langle \sum_i \mathsf{x}_i^*\otimes\mathsf{x}_i^*, \mathsf{a}\otimes\mathsf{b}\right\rangle,\end{aligned}$$

and the lemma follows. □

Proposition 7.12

$$\lim_{\hbar\to 0} \frac{\tilde{\mathbf{D}}_W - \mathsf{dim}\, W\, \mathsf{Id}}{\hbar^2\gamma_W} = \frac{1}{2}\Delta_{\mathfrak{h}^*} - \sum_{\alpha\in\mathsf{R}_+} \frac{\mathsf{e}_\alpha\mathsf{e}_{-\alpha}}{\left(e^{\langle\alpha,\lambda\rangle/2} - e^{-\langle\alpha,\lambda\rangle/2}\right)^2},$$

where $\Delta_{\mathfrak{h}^*} \stackrel{\text{def}}{=} \sum_i \partial_{\mathsf{x}_i^*}^2$.

Proof Lemma 7.11 implies that $\sum_{\nu\in\mathfrak{h}^*} \dim W[\nu] = \gamma_W \sum_i \partial^2_{\mathsf{x}_i^*} = \gamma_W \Delta_{\mathfrak{h}^*}$. Also, for $\alpha \in \mathsf{R}_+$, we have $\mathrm{tr}|_W\,(\mathsf{e}_{-\alpha}\mathsf{e}_\alpha) = \gamma_W \langle \mathsf{e}_{-\alpha}, \mathsf{e}_\alpha\rangle = \gamma_W$. Therefore,

$$\lim_{\hbar\to 0} \frac{\tilde{\mathbf{D}}_W - \dim W\,\mathrm{Id}}{\hbar^2\gamma_W}$$
$$= \lim_{\hbar\to 0}\frac{1}{\gamma_W}\left(\sum_{\nu\in\mathfrak{h}^*}\dim W[\nu]\frac{\partial_\nu^2}{2} - \sum_{\alpha\in\mathsf{R}_+}\left(\mathrm{tr}|_W\,\mathsf{e}_{-\alpha}\mathsf{e}_\alpha\right)\cdot\frac{\mathsf{e}_\alpha\mathsf{e}_{-\alpha}}{\left(e^{\langle\alpha,\lambda\rangle/2}-e^{-\langle\alpha,\lambda\rangle/2}\right)^2}\right)$$
$$= \frac{1}{2}\Delta_{\mathfrak{h}^*} - \sum_{\alpha\in\mathsf{R}_+}\frac{\mathsf{e}_\alpha\mathsf{e}_{-\alpha}}{\left(e^{\langle\alpha,\lambda\rangle/2}-e^{-\langle\alpha,\lambda\rangle/2}\right)^2}.$$

□

Definition 7.13 *The operator*

$$H = \frac{1}{2}\Delta_{\mathfrak{h}^*} - \sum_{\alpha\in\mathsf{R}_+}\frac{\mathsf{e}_\alpha\mathsf{e}_{-\alpha}}{\left(e^{\langle\alpha,\lambda\rangle/2}-e^{-\langle\alpha,\lambda\rangle/2}\right)^2} : \mathrm{Fun}(\mathfrak{h}^*, V[0]) \to \mathrm{Fun}(\mathfrak{h}^*, V[0])$$

is called the quantum spin Calogero–Moser Hamiltonian.

Remark 7.14 We know that the set $\{\mathbf{D}_W\}$ contains r algebraically independent commuting operators. However, Proposition 7.12 tells us that, as $\mathsf{q} \to 1$, $(\tilde{\mathbf{D}}_W - \dim W\,\mathrm{Id})/\hbar^2$ always tends to the same operator (up to a constant), regardless of what W is! However, it is possible to obtain algebraically independent operators that commute with the quantum spin Calogero–Moser Hamiltonian, by taking deeper terms of the expansion. For example,

- let W_1, W_2 be finite-dimensional representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$;
- let $a, b, c \in \mathbf{C}$ so that $a \dim W_1 + b \dim W_2 + c = 0$ and $a\gamma_{W_1} + b\gamma_{W_2} = 0$;
- let $\tilde{\mathbf{D}}^{(a,b,c)}_{W_1,W_2} = a\tilde{\mathbf{D}}_{W_1} + b\tilde{\mathbf{D}}_{W_2} + c\,\mathrm{Id}$.

Then the terms of orders $0, 1, 2$ in $\tilde{\mathbf{D}}^{(a,b,c)}_{W_1,W_2}$ are zero, while the nonzero term of lowest order in $\hbar$ will be an operator that commutes with the quantum spin Calogero–Moser Hamiltonian.

Example 7.15 (Usual Calogero–Moser model) Let $\mathfrak{g} = \mathfrak{sl}_n, k \in \mathbf{Z}_+$ and let V be the space of polynomials in $x_1, \ldots, x_n$ that are homogeneous of degree kn. There is a representation of $\mathfrak{g}$ on V: if $\alpha_{i,j} = e_i - e_j$, where e_k is the k^{th} standard basis vector, then $\mathsf{e}_\alpha = x_i(\partial/\partial x_j)$, while $\mathsf{h}_i = x_i(\partial/\partial x_i) - x_{i+1}(\partial/\partial x_{i+1})$. Then $V[0]$ is one-dimensional with basis $\{(x_1x_2\cdots x_n)^k\}$. We see that $(\mathsf{e}_\alpha\mathsf{e}_{-\alpha})|_{V[0]} = k(k+1)\mathrm{Id}$. Thus the spin Calogero–Moser operator is

$$H = \frac{1}{2}\sum_i^n\left(\frac{\partial}{\partial x_i}\right)^2 - \sum_{1\le i<j\le n}\frac{k(k+1)}{\left(e^{(x_i-x_j)/2}-e^{(x_j-x_i)/2}\right)^2}.$$

From a physical point of view, this describes a system of n quantum particles on the line, interacting with potential

$$\frac{k(k+1)}{\left(e^{(x-y)/2}-e^{(y-x)/2}\right)^2}, \quad \text{or} \quad \frac{k(k+1)}{4\sinh^2\left((x-y)/2\right)}.$$

Now recall that we have eigenfunctions $F_V(\lambda,\mu)$ for $\mathbf{D}_W$, with

$$\mathbf{D}_W^{(\lambda)}F_V(\lambda,\mu)=\chi_W(\mathsf{q}^{-2\bar{\mu}})F_V(\lambda,\mu).$$

Then $\tilde{F}_V(\lambda,\mu,\hbar)=F_V(\lambda/\hbar,\mu)$ is an eigenfunction for $\tilde{\mathbf{D}}_W$.

Proposition 7.16 *The following limit exists:*

$$F_V^c(\lambda,\mu)=\lim_{\hbar\to 0}\tilde{F}_V(\lambda,\mu,\hbar).$$

Proof In Section 7.2, we constructed $F_V(\lambda,\mu)$. If we use the same construction method on $\mathfrak{g}$ (instead of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$), we obtain $F_V^c(\lambda,\mu)$. □

Proposition 7.17 $HF_V^c(\lambda,\mu)=(\mu^2/2)F_V^c(\lambda,\mu)$, *and* $F_V^c(\lambda,\mu)$ *is an eigenfunction of the quantum integrals of* H *(operators that commute with* H*).*

Proof The first statement follows by taking the limit from the previous proposition. The second statement follows from the fact that for generic μ the multiplicity of the eigenvalue $\mu^2/2$ of the operator H in the space $e^{-\langle\lambda,\mu\rangle}\mathbf{C}[[e^{-\langle\lambda,\alpha_i\rangle}]]$ is 1. Thus any operator with coefficients in $\mathbf{C}[[e^{-\langle\lambda,\alpha_i\rangle}]]$ commuting with H must preserve the corresponding one-dimensional eigenspace. □

7.4 $F_V(\lambda,\mu)$ for $\mathfrak{sl}_2$

Before discussing the example of $\mathfrak{sl}_2$, let us prove two useful lemmas. These lemmas are well known and used widely in combinatorics. The second lemma is especially famous and goes under the name of the "q-binomial theorem".

Lemma 7.18 *Let* A *be an algebra, and let* $x,y\in A$ *be such that* $xy=pyx, p\in\mathbf{C}^*$. *Let* $\{a\}_p=(p^a-1)/(p-1)$ *and* $\{a\}_p!=\{1\}_p\cdots\{a\}_p$. *Then, for all* $k\geq 0$,

$$(x+y)^k=\sum_{l=0}^{k}\binom{k}{l}_p y^l x^{k-l},$$

where

$$\binom{k}{l}_p \stackrel{\text{def}}{=} \frac{\{k\}_p!}{\{l\}_p!\{k-l\}_p!}=\sum_{1\leq s_1<s_2\cdots<s_l\leq k} p^{\sum_{i=1}^{k}(s_i-i)}.$$

Proof The statement is obtained by straightforward multiplication, ordering factors, and collecting terms. □

Lemma 7.19 (the q-binomial theorem) *Let $l \geq 0$ be fixed. Then*

$$\sum_{k\geq l} \binom{k}{l}_p x^{k-l} = \prod_{i=0}^{l} (1-p^i x)^{-1}.$$

Proof Let y be such that $xy = pyx$. We have:

$$\begin{aligned}\sum_{k=0}^{\infty}(x+y)^k &= (1-x-y)^{-1}\\ &= (1-(1-x)^{-1}y)^{-1}(1-x)^{-1}\\ &= \sum_{l=0}^{\infty}((1-x)^{-1}y)^l(1-x)^{-1}\\ &= \sum_{l=0}^{\infty} y^l \prod_{i=0}^{l}(1-p^i x)^{-1}.\end{aligned}$$

So $\sum_{k=0}^{\infty}(x+y)^k = \sum_{l=0}^{\infty} y^l \prod_{i=0}^{l}(1-p^i x)^{-1}$, and the result follows upon extracting the coefficient of y^l and using Lemma 7.18. □

Example 7.20 Let $\mathfrak{g} = \mathfrak{sl}_2$, $V = V_{2m}, m \in \mathbf{Z}_+$. Then $V[0] = \mathbf{C}$, and so $\Psi_m \stackrel{\text{def}}{=} \Psi_V$ and $F_m \stackrel{\text{def}}{=} F_V$ are scalar-valued. We will compute Ψ_m.

We have a basis of M_μ, $\{\mathsf{F}^k \mathbf{v}_\mu\}$, and a basis of V, $\{w_\beta, \beta = m, m-1, \ldots, -m\}$, with $\mathsf{h}w_\beta = 2\beta w_\beta$ for all β, $\mathsf{F}w_\beta = w_{\beta-1}$ for $\beta \neq -m$, $\mathsf{F}w_{-m} = 0$. We write $\Phi_\mu^{w_0}\mathbf{v}_\mu = \sum_{j=0}^{m} c_j(\mu)\mathsf{F}^j\mathbf{v}_\mu \otimes w_j \in M_\mu \otimes V$. Then,

$$\begin{aligned}0 &= \Phi_\mu^{w_0}\mathsf{E}\mathbf{v}_\mu\\ &= (\boldsymbol{\Delta}\mathsf{E})\Phi_\mu^{w_0}\mathbf{v}_\mu\\ &= (\mathsf{E}\otimes \mathsf{q}^{\mathsf{h}} + 1\otimes \mathsf{E})\Phi_\mu^{w_0}\mathbf{v}_\mu\\ &= \sum_{j=0}^{m}(\mathsf{E}\otimes \mathsf{q}^{\mathsf{h}} + 1\otimes\mathsf{E})c_j(\mu)\mathsf{F}^j\mathbf{v}_\mu\otimes w_j\\ &= \sum_{j=0}^{m} c_j(\mu)\left(\mathsf{q}^{2j}\mathsf{E}\mathsf{F}^j\mathbf{v}_\mu\otimes w_j + \mathsf{F}^j\mathbf{v}_\mu\otimes\mathsf{E}w_j\right)\\ &= \sum_{j=1}^{m}\left(c_j(\mu)\mathsf{q}^{2j}[\mu-j+1]_{\mathsf{q}}[j]_{\mathsf{q}} + c_{j-1}(\mu)[m+j]_{\mathsf{q}}[m-j+1]_{\mathsf{q}}\right)\mathsf{F}^{j-1}\mathbf{v}_\mu\otimes w_j.\end{aligned}$$

So we have a recursive formula for $c_j(\mu)$:

$$c_j(\mu) = -c_{j-1}(\mu)\frac{[m+j]_{\mathsf{q}}[m-j+1]_{\mathsf{q}}}{\mathsf{q}^{2j}[\mu-j+1]_{\mathsf{q}}[j]_{\mathsf{q}}} \quad \text{for } j=1,\dots,m. \tag{7.1}$$

We also have $c_0(\mu)=1$, since $\langle \Phi_\mu^{w_0}\rangle = w_0$. So we can solve (7.1); we get

$$c_j(\mu) = (-1)^j \mathsf{q}^{-j(j+1)}\frac{[m+j]_{\mathsf{q}}!}{[j]_{\mathsf{q}}![m-j]_{\mathsf{q}}!}\cdot\prod_{i=1}^{j}[m-i+1]_{\mathsf{q}} \quad \text{for } j=0,\dots,m. \tag{7.2}$$

Now, for all k we have

$$\begin{aligned}
\Phi_\mu^{w_0}\mathsf{F}^k\mathbf{v}_\mu &= (\boldsymbol{\Delta}\mathsf{F})^k\Phi_\mu^{w_0}\mathbf{v}_\mu \\
&= (\mathsf{F}\otimes 1+\mathsf{q}^{-\mathsf{h}}\otimes\mathsf{F})^k\Phi_\mu^{w_0}\mathbf{v}_\mu \\
&= \sum_{l=0}^{k}\binom{k}{l}_{\mathsf{q}^{-2}}(\mathsf{q}^{-l\mathsf{h}}\mathsf{F}^{k-l}\otimes\mathsf{F}^l)\cdot\sum_{j=0}^{m}c_j(\mu)\mathsf{F}^j\mathbf{v}_\mu\otimes w_j \quad \text{by Lemma 7.18.}
\end{aligned}$$

Hence,

$$\begin{aligned}
\Psi_m(\lambda,\mu) &= (\mathsf{tr}|_{M_\mu}\otimes\mathsf{Id})\Phi_\mu^{w_0}\mathsf{q}^{2\bar\lambda}\otimes w_0^* \\
&= \sum_{k\geq 0}\sum_{l=0}^{\max\{k,m\}}\binom{k}{l}_{\mathsf{q}^{-2}}\mathsf{q}^{(\lambda-l)(\mu-2k)}c_l(\mu).
\end{aligned}$$

We would now like to compute an explicit expression for $F_V(\lambda,\mu)$. For this we need to compute $\mathbf{Q}(\mu)$. To do this, one can use Lemma 7.54. Namely, one should take the determinant of both sides of the equality in the lemma in a weight subspace of the tensor product of two finite-dimensional $\mathfrak{U}_{\mathsf{q}}(\mathfrak{sl}_2)$-modules (see Etingof and Varchenko (2000)). This yields recursive relations which allow one to compute $\mathbf{Q}(\mu)$ in the $\mathfrak{sl}_2$ case. We will omit this computation and give only the final result:

$$\mathbf{Q}(\mu)|_{V_{2m}[0]} = \mathsf{q}^{-2m}\prod_{j=1}^{m}\frac{\mathsf{q}^{-2\mu-2j+2}-\mathsf{q}^{-2m}}{\mathsf{q}^{-2\mu-2j}-1}.$$

Using this expression and the formula for $\Psi_m(\lambda,\mu)$, we obtain the desired formula for $F_V(\lambda,\mu)$:

$$\begin{aligned}
F_V(\lambda,\mu) = \ & \mathsf{q}^{-\lambda\mu}\prod_{j=1}^{m}\frac{\mathsf{q}^{-2\mu-2j}-1}{\mathsf{q}^{-2\mu-2j+2}-\mathsf{q}^{-2m}}\cdot\mathsf{q}^{2m}\sum_{l=0}^{m}\frac{[m+l]_{\mathsf{q}}!}{[l]_{\mathsf{q}}![m-l]_{\mathsf{q}}!} \\
&\cdot\frac{\mathsf{q}^{\frac{l(l-1)}{2}}(\mathsf{q}-\mathsf{q}^{-1})^l\mathsf{q}^{-2\lambda l}}{\prod_{j=1}^{l}(1-\mathsf{q}^{-2(\mu+j)})\prod_{j=1}^{l}(1-\mathsf{q}^{-2(\lambda-j)})}.
\end{aligned}$$

7.5 Center of $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$ and quantum traces

Let $\mathbf{H}$ be a Hopf algebra.

Notation 7.21 (Sweedler) *For $a \in \mathbf{H}$, we will write $\boldsymbol{\Delta}(a)$ as "$a_1 \otimes a_2$" or "$a^1 \otimes a^2$".*

Proposition 7.22 *Let $x \in \mathbf{H}$. Then the following are equivalent:*

1. *$a_1\, x\, a_2 = \epsilon(a)x$ for all $a \in \mathbf{H}$;*
2. *$x \in \mathsf{center}\,\mathbf{H}$.*

Proof For (2) $\Longrightarrow$ (1), suppose $x \in \mathsf{center}\,\mathbf{H}$. Then

$$a_1\, x\, \mathbf{S}(a_2) = x \sum_i a_1 \mathbf{S}(a_2) = x\epsilon(a) = \epsilon(a)x.$$

For (1) $\Longrightarrow$ (2), suppose $a_1\, x\, \mathbf{S}(a_2) = \epsilon(a)x$ for all $a \in \mathbf{H}$. Let $b \in \mathbf{H}$. Then

$$\begin{aligned}
xb &= x(\epsilon \otimes \mathsf{Id})\boldsymbol{\Delta}(b)\\
&= x\epsilon(b_1)b_2\\
&= b_{11}\, x\, \mathbf{S}(b_{12})b_2\\
&= \mathbf{m}(\mathbf{m} \otimes \mathsf{Id})(\mathsf{Id} \otimes x\,\mathbf{S} \otimes \mathsf{Id})((\boldsymbol{\Delta} \otimes \mathsf{Id})\boldsymbol{\Delta}(b))\\
&= \mathbf{m}(\mathbf{m} \otimes \mathsf{Id})(\mathsf{Id} \otimes x\,\mathbf{S} \otimes \mathsf{Id})((\mathsf{Id} \otimes \boldsymbol{\Delta})\boldsymbol{\Delta}(b))\\
&= b_1\, x\, \mathbf{S}(b_{21})b_{22}\\
&= b_1\, x\, \epsilon(b_2)\\
&= b_1\epsilon(b_2)x\\
&= (\mathsf{Id} \otimes \epsilon)\boldsymbol{\Delta}(b)x\\
&= bx.
\end{aligned}$$

□

This proposition is easily generalized.

Definition 7.23 *Let H be an algebra. An H-bimodule is a vector space V together with two actions, $m_1 : H \otimes V \to V$ (denoted $h \otimes v \mapsto hv$) and $m_2 : V \otimes H \to V$ (denoted $v \otimes h \mapsto vh$), such that $(h_1 v)h_2 = h_1(vh_2)$. Clearly, an H-bimodule is the same thing as an $H \otimes H^{op}$-module, where H^{op} is the algebra H with opposite multiplication.*

Proposition 7.24 *Let V be an $\mathbf{H}$-bimodule, and let $v \in V$. Then the following are equivalent:*

1. *$a_1\, v\, \mathbf{S}(a_2) = \epsilon(a)v$ for all $a \in \mathbf{H}$;*
2. *$va = av$ for all $a \in \mathbf{H}$.*

Definition 7.25 *A linear functional* $\theta : \mathbf{H} \to \mathbf{C}$ *such that* $\theta(xy) = \theta(y\mathbf{S}^2(x))$ *for all* $x, y \in \mathbf{H}$ *is called a quantum trace.*

Example 7.26 Suppose V is a finite-dimensional representation of $\mathbf{H}$ and $g \in \mathbf{H}$ is such that $gxg^{-1} = \mathbf{S}^2(x)$ for all $x \in \mathbf{H}$. Then $\theta(a) \stackrel{\text{def}}{=} \mathsf{tr}|_V\,(ag)$ is a quantum trace, since $\theta(xy) = \mathsf{tr}|_V\,(xyg) = \mathsf{tr}|_V\,(ygx) = \mathsf{tr}|_V\,(y\mathbf{S}^2(x)g) = \theta(y\mathbf{S}^2(x))$.

Proposition 7.27 (Drinfeld (1990*a*), Reshetikhin (1990)) *Let* $\theta : \mathbf{H} \to \mathbf{C}$ *be a quantum trace. Suppose that* $z \in \mathbf{H} \otimes \mathbf{H}$ *is such that* $(\boldsymbol{\Delta}(a))z = z(\boldsymbol{\Delta}(a))$ *for all* $a \in \mathbf{H}$*. Then* $C = (\mathsf{Id} \otimes \theta)z \in \mathsf{center}\,\mathbf{H}$*.*

Lemma 7.28

$$(\boldsymbol{\Delta} \otimes \boldsymbol{\Delta})\boldsymbol{\Delta} = (\mathsf{Id} \otimes \boldsymbol{\Delta} \otimes \mathsf{Id})(\boldsymbol{\Delta} \otimes \mathsf{Id})\boldsymbol{\Delta}.$$

Proof $(\boldsymbol{\Delta} \otimes \boldsymbol{\Delta})\boldsymbol{\Delta} = (\boldsymbol{\Delta} \otimes \mathsf{Id} \otimes \mathsf{Id})(\mathsf{Id} \otimes \boldsymbol{\Delta})\boldsymbol{\Delta} = (\boldsymbol{\Delta} \otimes \mathsf{Id} \otimes \mathsf{Id})(\boldsymbol{\Delta} \otimes \mathsf{Id})\boldsymbol{\Delta} = (\mathsf{Id} \otimes \boldsymbol{\Delta} \otimes \mathsf{Id})(\boldsymbol{\Delta} \otimes \mathsf{Id})\boldsymbol{\Delta}$. □

Proof of Proposition 7.27 Let $V = \mathbf{H} \otimes \mathbf{H}$, with $a \otimes v \mapsto \boldsymbol{\Delta}(a)v$ and $v \otimes a \mapsto v\boldsymbol{\Delta}(a)$. Then, for all $a \in \mathbf{H}$, we have $az = za$. By Proposition 7.24, this implies $a_1\, z\, \mathbf{S}(a_2) = \epsilon(a)z$. Hence

$$\begin{aligned}
\epsilon(a)z &= \boldsymbol{\Delta}(a_1)\, z\, \boldsymbol{\Delta}(\mathbf{S}(a_2)) \\
&= \boldsymbol{\Delta}(a_1)\, z\, (\mathbf{S} \otimes \mathbf{S})\boldsymbol{\Delta}^{\text{op}}(a_2) \\
&= \mathbf{m}(\mathbf{m} \otimes \mathbf{m})((\mathsf{Id} \otimes \mathsf{Id})^{12}\, z\, (\mathbf{S} \otimes \mathbf{S})^{43})(\boldsymbol{\Delta} \otimes \boldsymbol{\Delta})\boldsymbol{\Delta}(a) \\
&= \mathbf{m}(\mathbf{m} \otimes \mathbf{m})((\mathsf{Id} \otimes \mathsf{Id})^{12}\, z\, (\mathbf{S} \otimes \mathbf{S})^{43})(\mathsf{Id} \otimes \boldsymbol{\Delta} \otimes \mathsf{Id})(\boldsymbol{\Delta} \otimes \mathsf{Id})\boldsymbol{\Delta}(a) \\
&\qquad \text{(by Lemma 7.28)} \\
&= (a_{11} \otimes a_{121})\, z\, (\mathbf{S}(a_2) \otimes \mathbf{S}(a_{122})). \qquad (7.3)
\end{aligned}$$

Applying $(\mathsf{Id} \otimes \theta)$ to both sides of (7.3), we get

$$\begin{aligned}
C\epsilon(a) &= (\mathsf{Id} \otimes \theta)((a_{11} \otimes a_{121})\, z\, (\mathbf{S}(a_2) \otimes \mathbf{S}(a_{122}))) \\
&= (\mathsf{Id} \otimes \theta)((a_{11} \otimes \mathsf{Id})\, z\, (\mathbf{S}(a_2) \otimes \mathbf{S}(a_{122})\mathbf{S}^2(a_{121}))) \quad (\theta \text{ is a quantum trace}) \\
&= (\mathsf{Id} \otimes \theta)((a_{11} \otimes \mathsf{Id})\, z\, (\mathbf{S}(a_2) \otimes \epsilon(a_{12})\, \mathsf{Id})) \quad (\text{since } \mathbf{Sm}(\mathbf{S} \otimes \mathsf{Id})\boldsymbol{\Delta} = \mathbf{S}\epsilon = \epsilon) \\
&= (\mathsf{Id} \otimes \theta)((a_1 \otimes \mathsf{Id})\, z\, (\mathbf{S}(a_2) \otimes \mathsf{Id})) \\
&= a_1(\mathsf{Id} \otimes \theta)(z)\mathbf{S}(a_2) \\
&= a_1 C\mathbf{S}(a_2).
\end{aligned}$$

By Proposition 7.24, this implies that $C \in \mathsf{center}\,\mathbf{H}$. □

Corollary 7.29 *Let $g \in \mathbf{H}$ be such that $gxg^{-1} = \mathbf{S}^2(x)$ for all $x \in \mathbf{H}$. Let $z \in \mathbf{H} \otimes \mathbf{H}$ be such that $\boldsymbol{\Delta}(a)z = z\boldsymbol{\Delta}a$ for all $a \in \mathbf{H}$. Then, for any finite-dimensional representation V of $\mathbf{H}$, we have $(\mathsf{Id} \otimes \mathsf{tr}|_V)(z(1 \otimes g)) \in \mathsf{center}\,\mathbf{H}$.*

Proof We know from Example 7.26 that $\theta(a) \stackrel{\text{def}}{=} \mathsf{tr}|_V\,(ag)$ is a quantum trace, and the result follows from Proposition 7.27. □

Corollary 7.30 (Drinfeld (1990*a*), Reshetikhin (1990)) *Let V be a finite-dimensional representation of $\mathfrak{U}_q(\mathfrak{g})$. Then*

$$C_V \stackrel{\text{def}}{=} (\mathsf{Id} \otimes \mathsf{tr}|_V)(\mathcal{R}^{21}\mathcal{R}(1 \otimes \mathsf{q}^{2\bar\rho}))$$

belongs to $\mathfrak{U}_q(\mathfrak{g})$ and is central.

Proof We apply Proposition 7.29 with $\mathbf{H} = \mathfrak{U}_q(\mathfrak{g}), z = \mathcal{R}^{21}\mathcal{R}$ and $g = \mathsf{q}^{2\bar\rho}$. All we need to do is show that $C_V \in \mathfrak{U}_q(\mathfrak{g})$, i.e. that it is a *finite* sum. It is enough to show that for $v \in V, f \in V^*$, $(\mathsf{Id} \otimes f)(\mathcal{R}^{21}\mathcal{R}(1 \otimes v))$ is a finite sum, which is easy. □

Example 7.31 Let $\mathfrak{g} = \mathfrak{sl}_2, V = \mathbf{C}^2 = \mathsf{span}\{v_+, v_-\}$. We will calculate C_V. We see that

$$\begin{aligned} C_V &= (\mathsf{Id} \otimes \mathsf{tr}|_V)(\mathcal{R}^{21}\mathcal{R}(\mathsf{Id} \otimes \mathsf{q}^{2\bar\rho})) \\ &= (\mathsf{Id} \otimes \mathsf{tr}|_V)(\mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{Id} + (\mathsf{q} - \mathsf{q}^{-1})\mathsf{F} \otimes \mathsf{E})\mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{Id} + (\mathsf{q} - \mathsf{q}^{-1})\mathsf{E} \otimes \mathsf{F})(\mathsf{Id} \otimes \mathsf{q}^{\mathsf{h}})) \\ &\qquad (\text{since } \mathsf{E}^2 V = \mathsf{F}^2 V = 0) \\ &= (\mathsf{Id} \otimes \mathsf{tr}|_V)(\mathsf{q}^{\mathsf{h}\otimes\mathsf{h}} + (\mathsf{q} - \mathsf{q}^{-1})^2 \mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{F} \otimes \mathsf{E})\mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{E} \otimes \mathsf{F})(\mathsf{Id} \otimes \mathsf{q}^{\mathsf{h}})) \\ &\qquad \text{by triangularity.} \end{aligned}$$

Now

$$\begin{aligned} \mathsf{q}^{\mathsf{h}\otimes\mathsf{h}}(\mathsf{Id} \otimes \mathsf{q}^{\mathsf{h}})(\mathsf{Id} \otimes v_+) &= \mathsf{q}^{\mathsf{h}+1} \otimes v_+; \\ \mathsf{q}^{\mathsf{h}\otimes\mathsf{h}}(\mathsf{Id} \otimes \mathsf{q}^{\mathsf{h}})(\mathsf{Id} \otimes v_-) &= \mathsf{q}^{-\mathsf{h}-1} \otimes v_-; \\ \mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{F} \otimes \mathsf{E})\mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{E} \otimes \mathsf{F})(\mathsf{Id} \otimes \mathsf{q}^{\mathsf{h}})(\mathsf{Id} \otimes v_+) &= (\mathsf{q}\mathsf{q}^{\frac{\mathsf{h}}{2}}\mathsf{F}\mathsf{q}^{\frac{-\mathsf{h}}{2}}\mathsf{E} \otimes v_+) \\ &= \mathsf{FE} \otimes v_+; \\ \mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{F} \otimes \mathsf{E})\mathsf{q}^{\frac{\mathsf{h}\otimes\mathsf{h}}{2}}(\mathsf{E} \otimes \mathsf{F})(\mathsf{Id} \otimes \mathsf{q}^{\mathsf{h}})(\mathsf{Id} \otimes v_-) &= 0. \end{aligned}$$

Hence,

$$C_V = \mathsf{q}^{\mathsf{h}+1} + \mathsf{q}^{-\mathsf{h}-1} + (\mathsf{q} - \mathsf{q}^{-1})^2\mathsf{FE}.$$

Note that this is true for $\mathsf{q} \neq 1$. For $\mathsf{q} = 1$, we have $C_V = 2\,\mathsf{Id}$. Nevertheless, we can recover the Casimir even for $\mathsf{q} = 1$ by taking the limit of $(C_V - 2\,\mathsf{Id})/(\mathsf{q} - \mathsf{q}^{-1})^2$ as $\mathsf{q} \to 1$.

Theorem 7.32 *The assignment* $V \mapsto C_V$ *is a ring homomorphism*

$$\mathcal{G}(\mathsf{Rep}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})) \to \mathsf{center}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g});$$

in other words, $C_{V\otimes W} = C_V C_W$ *for all* $V, W \in \mathsf{Rep}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$.

Proof

$$\begin{aligned}
C_{V\otimes W} &= (\mathsf{Id}\otimes \mathsf{tr}|_V \otimes \mathsf{tr}|_W)(((\mathsf{Id}\otimes\boldsymbol{\Delta})\mathcal{R}^{21})((\mathsf{Id}\otimes\boldsymbol{\Delta})\mathcal{R})(1\otimes \mathsf{q}^{2\bar\rho}\otimes \mathsf{q}^{2\bar\rho}))\\
&= (\mathsf{Id}\otimes \mathsf{tr}|_V \otimes \mathsf{tr}|_W)(\mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12}(1\otimes \mathsf{q}^{2\bar\rho}\otimes \mathsf{q}^{2\bar\rho}))\\
&= (\mathsf{Id}\otimes \mathsf{tr}|_V)(\mathcal{R}^{21}(C_W\otimes 1)\mathcal{R}^{12}(1\otimes \mathsf{q}^{2\bar\rho}))\\
&= C_W(\mathsf{Id}\otimes \mathsf{tr}|_V)(\mathcal{R}^{21}\mathcal{R}^{12}(1\otimes \mathsf{q}^{2\bar\rho})) \quad \text{(since } C_W \text{ is central)}\\
&= C_V C_W.
\end{aligned}$$

□

Remark 7.33 If q is not a root of unity, then the homomorphism of Theorem 7.32 induces an isomorphism $\mathcal{G}(\mathsf{Rep}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g}))\otimes \mathbf{C} \cong \mathsf{center}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$. If $\mathsf{q} = 1$, then it does *not* induce such an isomorphism; nevertheless, $\mathcal{G}(\mathsf{Rep}\,\mathfrak{U}(\mathfrak{g}))\otimes \mathbf{C}$ and $\mathsf{center}\,\mathfrak{U}(\mathfrak{g})$ are still isomorphic. Namely, we have $\mathcal{G}(\mathsf{Rep}\,\mathfrak{U}(\mathfrak{g}))\otimes \mathbf{C} \cong \mathbf{C}[x_1,\ldots,x_r]$, where $r = \mathsf{rank}\,\mathfrak{g}$, and we also have $\mathsf{center}\,\mathfrak{U}_{\mathsf{q}}(\mathfrak{g}) \cong \mathbf{C}[x_1,\ldots,x_r]$ through the Harish–Chandra isomorphism.

Theorem 7.34

$$C_V|_{M_\mu} = \chi_V(\mathsf{q}^{2(\bar\mu+\bar\rho)})\,\mathsf{Id}\,.$$

Proof We see that

$$\begin{aligned}
C_V\mathbf{v}_\mu &= (\mathsf{Id}\otimes \mathsf{tr}|_V)(\mathcal{R}^{21}\mathcal{R}(1\otimes \mathsf{q}^{2\bar\rho}))\mathbf{v}_\mu\\
&= (\mathsf{Id}\otimes \mathsf{tr}|_V)(\mathcal{R}^{21}\mathsf{q}^{\sum_i \mathsf{x}_i\otimes \mathsf{x}_i}(1\otimes \mathsf{q}^{2\bar\rho}))\mathbf{v}_\mu\\
&= (\mathsf{Id}\otimes \mathsf{tr}|_V)(\mathsf{q}^{2\sum_i \mathsf{x}_i\otimes \mathsf{x}_i}(1\otimes \mathsf{q}^{2\bar\rho}))\mathbf{v}_\mu\\
&= (\mathsf{tr}|_V\,(\mathsf{q}^{2\sum_i \mu(\mathsf{x}_i)\mathsf{x}_i}\mathsf{q}^{2\bar\rho}))\mathbf{v}_\mu\\
&= (\mathsf{tr}|_V\,\mathsf{q}^{2\bar\mu+2\bar\rho})\mathbf{v}_\mu\\
&= \chi_V(\mathsf{q}^{2\bar\mu+2\bar\rho})\mathbf{v}_\mu.
\end{aligned}$$

For all $a \in \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$, we have $C_V(a\mathbf{v}_\mu) = aC_V\mathbf{v}_\mu = \chi_V(\mathsf{q}^{2\bar\mu+2\bar\rho})(a\mathbf{v}_\mu)$ since C_V is central. So $C_V|_{M_\mu} = \chi_V(\mathsf{q}^{2\bar\mu+2\bar\rho})\,\mathsf{Id}$. □

7.6 The functions Z_V and X_V

Now we begin the proof of Theorem 7.7. We will first prove the Macdonald-Ruijsenaars equations for F_V. The proof depends on a series of lemmas.

We will be working in $M_\mu \otimes V \otimes V^* \otimes \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$. The four components of this tensor product will be respectively labelled $0, 1, 1^*$ and 2, and subscripts or superscripts $0, 1, 1^*, 2$ or a combination of them will indicate that the corresponding expression lives in the specified components. Let

$$\Phi_\mu^V = \sum_i \Phi_\mu^{v_i} \otimes v_i^* : M_\mu \to M_\mu \otimes V \otimes V^*,$$

where $\{v_i\}$ is a basis of V. Let

$$Z_V(\lambda, \mu) = \mathsf{tr}|_0 \, (\Phi_\mu^{V,011^*} \mathcal{R}^{20} \mathsf{q}_0^{2\bar{\lambda}}).$$

Lemma 7.35 *Then*

$$Z_V(\lambda, \mu) = \mathcal{R}^{21} \mathsf{q}_1^{2\bar{\lambda}} Z_V(\lambda, \mu).$$

Proof

$$\begin{aligned}
Z_V(\lambda, \mu) &= \mathsf{tr}|_0 \, (\Phi_\mu^{V,011^*} \mathcal{R}^{20} \mathsf{q}_0^{2\bar{\lambda}}) \\
&= \mathsf{tr}|_0 \, (((\mathsf{Id} \otimes \boldsymbol{\Delta})(\mathcal{R}))^{201} \Phi_\mu^{V,011^*} \mathsf{q}_0^{2\bar{\lambda}}) \\
&= \mathsf{tr}|_0 \, (\mathcal{R}^{21} \mathcal{R}^{20} \Phi_\mu^{V,011^*} \mathsf{q}_0^{2\bar{\lambda}}) \\
&= \mathcal{R}^{21} \ \mathsf{tr}|_0 \, (\mathcal{R}^{20} \Phi_\mu^{V,011^*} \mathsf{q}_0^{2\bar{\lambda}}) \\
&= \mathcal{R}^{21} \ \mathsf{tr}|_0 \, (\Phi_\mu^{V,011^*} \mathsf{q}_0^{2\bar{\lambda}} \mathcal{R}^{20}) \\
&= \mathcal{R}^{21} \ \mathsf{tr}|_0 \, (\mathsf{q}_0^{2\bar{\lambda}} \mathsf{q}_1^{2\bar{\lambda}} \Phi_\mu^{V,011^*} \mathcal{R}^{20}) \\
&= \mathcal{R}^{21} \mathsf{q}_1^{2\bar{\lambda}} \ \mathsf{tr}|_0 \, (\Phi_\mu^{V,011^*} \mathcal{R}^{20} \mathsf{q}_0^{2\bar{\lambda}}) \\
&= \mathcal{R}^{21} \mathsf{q}_1^{2\bar{\lambda}} Z_V(\lambda, \mu).
\end{aligned}$$

□

Definition 7.36 *Let J be the usual (universal) fusion operator. Then the modified fusion operator is $\mathcal{J}(\lambda) = J(-\lambda - \rho + \frac{1}{2}(h^1 + h^2))$.*

Lemma 7.37

$$\mathcal{R}^{21} (\mathsf{q}^{2\bar{\lambda}})_1 \mathcal{J}(\lambda) = \mathcal{J}(\lambda) \mathsf{q}^{\sum_i \mathsf{x}_i \otimes \mathsf{x}_i} (\mathsf{q}^{2\bar{\lambda}})_1.$$

Proof Let W, V be representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$. We know from the ABRR equation that

$$J_{WV}(\lambda)\left(\mathsf{Id}\otimes\mathsf{q}^{2(\bar\lambda+\bar\rho)-\sum_i \mathsf{x}_i^2}\right) = \mathcal{R}^{21}_{VW}\mathsf{q}^{-\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\left(\mathsf{Id}\otimes\mathsf{q}^{2(\bar\lambda+\bar\rho)-\sum_i \mathsf{x}_i^2}\right)J_{WV}(\lambda). \tag{7.4}$$

Now we right-multiply both sides of (7.4) by $\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{q}^{-2(\bar\lambda+\bar\rho)}$. Since J_{WV} has zero weight, we know that it commutes with $\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{q}^{-2(\bar\lambda+\bar\rho)}$. Thus,

$$\begin{aligned} &J_{WV}(\lambda)\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{q}^{-\sum_i \mathsf{x}_i^2}\right)\\ &= \mathcal{R}^{21}_{VW}\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{Id}\right)\mathsf{q}^{-\sum_i(\mathsf{x}_i\otimes\mathsf{x}_i+\mathsf{Id}\otimes\mathsf{x}_i^2)}J_{WV}(\lambda). \end{aligned} \tag{7.5}$$

Now, if $w\otimes v\in W[\nu-\beta]\otimes V[\mu+\beta]$, then

$$\begin{aligned} \mathsf{q}^{-\sum_i(\mathsf{x}_i\otimes\mathsf{x}_i+\mathsf{Id}\otimes\mathsf{x}_i^2)}(w\otimes v) &= \mathsf{q}^{-(\langle\nu-\beta,\mu+\beta\rangle+\langle\mu+\beta,\mu+\beta\rangle)}(w\otimes v)\\ &= \mathsf{q}^{-\langle\nu+\mu,\mu+\beta\rangle}(w\otimes v)\\ &= (\mathsf{Id}\otimes\mathsf{q}^{-(\bar\nu+\bar\mu)})(w\otimes v). \end{aligned}$$

Let us apply both sides of (7.5) to $W[\nu]\otimes V[\mu]$. This gives

$$\begin{aligned} &J_{WV}(\lambda)\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{q}^{-\bar\mu}\right)\\ &= \mathcal{R}^{21}_{VW}\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{q}^{-(\bar\nu+\bar\mu)}\right)J_{WV}(\lambda)\quad (\text{on } W[\nu]\otimes V[\mu]). \end{aligned} \tag{7.6}$$

Right-multiplying both sides of (7.6) by $\mathsf{q}^{(\bar\nu+\bar\mu)}\otimes\mathsf{q}^{(\bar\nu+\bar\mu)}$ (using again the fact that J has zero weight) gives

$$\begin{aligned} &J_{WV}(\lambda)\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)-(\bar\nu+\bar\mu)}\otimes\mathsf{q}^{\bar\nu}\right)\\ &= \mathcal{R}^{21}_{VW}\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)-(\bar\nu+\bar\mu)}\otimes\mathsf{Id}\right)J_{WV}(\lambda)\quad (\text{on } W[\nu]\otimes V[\mu]). \end{aligned} \tag{7.7}$$

We then replace λ by $\lambda+\frac{1}{2}(\nu+\mu)$ in (7.7). We get:

$$\begin{aligned} &J_{WV}\left(\lambda+\tfrac{1}{2}(\nu+\mu)\right)\left(\mathsf{q}^{-2(\bar\lambda+\bar\rho)}\otimes\mathsf{q}^{\bar\nu}\right)\\ &= \mathcal{R}^{21}_{VW}\mathsf{q}_1^{-2(\bar\lambda+\bar\rho)}J_{WV}\left(\lambda+\tfrac{1}{2}(\nu+\mu)\right)\quad (\text{on } W[\nu]\otimes V[\mu]). \end{aligned}$$

Thus the following holds on $W\otimes V$:

$$\begin{aligned} &J_{WV}\left(\lambda+\tfrac{1}{2}(h^1+h^2)\right)\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\mathsf{q}_1^{-2(\bar\lambda+\bar\rho)}\\ &= \mathcal{R}^{21}_{VW}\mathsf{q}_1^{-2(\bar\lambda+\bar\rho)}J_{WV}\left(\lambda+\tfrac{1}{2}(h^1+h^2)\right). \end{aligned}$$

The result follows upon replacing λ by $-(\lambda+\rho)$. □

Lemma 7.38

$$\mathcal{J}^{12,3}(\lambda)\mathcal{J}^{12}\left(\lambda+\tfrac{1}{2}h^3\right)=\mathcal{J}^{1,23}(\lambda)\mathcal{J}^{23}\left(\lambda-\tfrac{1}{2}h^1\right).$$

Proof We know that

$$J^{12,3}(\lambda)J^{12}(\lambda-h^3)=J^{1,23}(\lambda)J^{23}(\lambda). \tag{7.8}$$

Using the fact that J has zero weight, we can replace λ by $-\lambda-\rho+\frac{1}{2}(h^1+h^2+h^3)$ in (7.8), and the result follows. □

Lemma 7.39 *For generic λ, a solution*

$$X=\sum_{\beta\in \mathsf{Q}_+}x^{(\beta)}, \mathsf{wt}^{12}\,x^{(\beta)}=(-\beta,\beta) \tag{7.9}$$

of the equation

$$X=\mathcal{R}^{21}\mathsf{q}_1^{2\bar{\lambda}}X \tag{7.10}$$

is uniquely determined by $x^{(0)}$ (here wt^{12} denotes the weight in the first and second component).

Proof We may write

$$\mathcal{R}=\sum_{\gamma\in\mathsf{Q}_+}L'^{(\gamma)},\quad \mathsf{wt}^{12}\,L'^{(\gamma)}=(\gamma,-\gamma),\quad L'^{(0)}=\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}.$$

So if (7.9) is a solution of (7.10), then

$$\sum_{\beta\in\mathsf{Q}_+}x^{(\beta)}=\sum_{\gamma,\beta\in\mathsf{Q}_+}\mathsf{q}^{-2\langle\lambda,\beta\rangle}(L'^{(\gamma)})^{21}x^{(\beta)}. \tag{7.11}$$

We then extract the term of weight $(-\beta',\beta')$ from both sides of (7.11). Using the fact that $(L'^{(0)})^{21}x^{(\beta')}=\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}x^{(\beta')}=\mathsf{q}^{\left\langle\beta',\beta'\right\rangle}x^{(\beta')}$, this gives us

$$x^{(\beta')}=\sum_{0\le\beta<\beta'}\mathsf{q}^{-2\langle\lambda,\beta\rangle}(L'^{(\beta'-\beta)})^{21}x^{(\beta)}+\mathsf{q}^{-\left\langle 2\lambda+\beta',\beta'\right\rangle}x^{(\beta')}. \tag{7.12}$$

For generic λ, we have $\langle 2\lambda+\beta',\beta'\rangle\neq 0$ for whenever $\beta'\in\mathsf{Q}_+\setminus\{0\}$; thus, (7.12) becomes

$$x^{(\beta')}=\frac{1}{1-\mathsf{q}^{-\langle 2\lambda+\beta',\beta'\rangle}}\sum_{0\le\beta<\beta'}\mathsf{q}^{-2\langle\lambda,\beta\rangle}(L'^{(\beta'-\beta)})^{21}x^{(\beta)},\quad \text{for all } \beta'\in\mathsf{Q}_+\setminus\{0\}.$$

By induction, it follows that X is uniquely determined by $x^{(0)}$. □

Proposition 7.40

$$Z_V(\lambda,\mu) = \mathcal{J}^{12}(\lambda)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right).$$

Proof On the one hand, we see that $Z = Z_V(\lambda,\mu)$ satisfies (7.10), by Lemma 7.35. On the other hand, Lemma 7.37 implies that

$$\mathcal{R}^{21}(\mathsf{q}^{2\bar\lambda})_1\mathcal{J}(\lambda)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right) = \mathcal{J}(\lambda)\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}(\mathsf{q}^{2\bar\lambda})_1\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right). \quad (7.13)$$

But $\Psi_V \in V[0]\otimes V^*[0]$, so $(\mathfrak{h}\otimes\mathfrak{h})\Psi_V = 0$. Thus (7.13) becomes

$$\mathcal{R}^{21}(\mathsf{q}^{2\bar\lambda})_1\mathcal{J}^{12}(\lambda)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right) = \mathcal{J}^{12}(\lambda)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right),$$

hence $Y = \mathcal{J}^{12}(\lambda)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right)$ also satisfies (7.10).

Furthermore, the term of weight $(0,0)$ in Y is $\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right)$. Let us now compute the term of the same weight in Z. We note that $(Z_V(\lambda,\mu))^{(0)} = \mathsf{tr}|_0\,(\Phi_\mu^{V,011^*}(\mathcal{R}^{20})^{(0)}\mathsf{q}_0^{2\bar\lambda}) = \mathsf{tr}|_0\,(\Phi_\mu^{V,011^*}\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\mathsf{q}_0^{2\bar\lambda})$. Then, let $v \in M_\mu, y \in \mathfrak{U}_\mathsf{q}(\mathfrak{g})$. We have:

$$(\Phi_\mu^{V,011^*}\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\mathsf{q}_0^{2\bar\lambda})(v\otimes y) = (\Phi_\mu^{V,011^*}\mathsf{q}_0^{2(\frac{1}{2}\overline{\mathsf{wt}\,y}+\bar\lambda)})(v\otimes y)$$
$$\Longrightarrow \mathsf{tr}|_0\,(\Phi_\mu^{V,011^*}\mathsf{q}^{\sum_i \mathsf{x}_i\otimes\mathsf{x}_i}\mathsf{q}_0^{2\bar\lambda}) = \Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right);$$

hence, $Z^{(0)} = Y^{(0)} = \Psi_V^1\left(\lambda + \tfrac{1}{2}h^2, \mu\right)$. By Lemma 7.39, we conclude that $Z = Y$, which is the desired result. □

We will now be working in $M_\mu\otimes V\otimes V^*\otimes\mathfrak{U}_\mathsf{q}(\mathfrak{g})\otimes\mathfrak{U}_\mathsf{q}(\mathfrak{g})$. The five components of this tensor product will be, respectively, labelled $0, 1, 1^*, 2$ and 3. Let $X_V(\lambda,\mu) = \mathsf{tr}|_0\,(\Phi_\mu^{011^*}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda}(\mathcal{R}^{03})^{-1})$.

Lemma 7.41

$$X_V(\lambda,\mu) = \mathcal{R}^{12,3}\mathsf{q}_3^{2\bar\lambda}X_V(\lambda,\mu)\mathsf{q}_3^{-2\bar\lambda}(\mathcal{R}^{23})^{-1}.$$

Proof

$$\begin{aligned}
X_V(\lambda,\mu) &= \mathsf{tr}|_0\,(\Phi_\mu^{011^*}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda}(\mathcal{R}^{03})^{-1})\\
&= \mathsf{tr}|_0\,((\mathcal{R}^{03})^{-1}\Phi_\mu^{011^*}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda})\\
&= \mathcal{R}^{13}\ \mathsf{tr}|_0\,(\Phi_\mu^{011^*}(\mathcal{R}^{03})^{-1}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda})\\
&= \mathcal{R}^{13}\ \mathsf{tr}|_0\,(\Phi_\mu^{011^*}\mathcal{R}^{23}\mathcal{R}^{20}(\mathcal{R}^{03})^{-1}(\mathcal{R}^{23})^{-1}\mathsf{q}_0^{2\bar\lambda})\\
&= \mathcal{R}^{13}\mathcal{R}^{23}\ \mathsf{tr}|_0\,(\Phi_\mu^{011^*}\mathcal{R}^{20}(\mathcal{R}^{03})^{-1}\mathsf{q}_0^{2\bar\lambda})(\mathcal{R}^{23})^{-1}\\
&= \mathcal{R}^{12,3}\mathsf{q}_3^{2\bar\lambda}X_V(\lambda,\mu)\mathsf{q}_3^{-2\bar\lambda}(\mathcal{R}^{23})^{-1}.
\end{aligned}$$

□

Lemma 7.42 *Let $Y(\lambda,\mu) = \mathcal{J}^{3,12}(\lambda)^{-1} X_V(\lambda,\mu)\mathcal{J}^{32}(\lambda)$. Then,*

$$\mathsf{q}_3^{2\bar\lambda}\mathsf{q}^{\sum_i(\mathsf{x}_i\otimes 1\otimes \mathsf{x}_i + 1\otimes\mathsf{x}_i\otimes\mathsf{x}_i)} Y(\lambda,\mu) = Y(\lambda,\mu)\mathsf{q}_3^{2\bar\lambda}\mathsf{q}^{\sum_i 1\otimes\mathsf{x}_i\otimes\mathsf{x}_i}. \tag{7.14}$$

Proof We have

$$\begin{aligned}
Y(\lambda,\mu) &= \mathcal{J}^{3,12}(\lambda)^{-1}\mathsf{q}_3^{-2\bar\lambda}(\mathcal{R}^{12,3})^{-1}X_V(\lambda,\mu)\mathcal{R}^{23}\mathsf{q}_3^{2\bar\lambda}\mathcal{J}^{32}(\lambda) \quad \text{by Lemma 7.41}\\
&= \left((\mathcal{R}^{12,3})\mathsf{q}_3^{2\bar\lambda}\mathcal{J}^{3,12}(\lambda)\right)^{-1} X_V(\lambda,\mu)\mathcal{R}^{23}\mathsf{q}_3^{2\bar\lambda}\mathcal{J}^{32}(\lambda)\\
&= \left(\mathcal{J}^{3,12}(\lambda)\mathsf{q}^{\sum_i(\mathsf{x}_i\otimes 1\otimes\mathsf{x}_i+1\otimes\mathsf{x}_i\otimes\mathsf{x}_i)}\mathsf{q}_3^{2\bar\lambda}\right)^{-1} X_V(\lambda,\mu)\mathcal{J}^{32}(\lambda)\mathsf{q}^{\sum_i 1\otimes\mathsf{x}_i\otimes\mathsf{x}_i}\mathsf{q}_3^{2\bar\lambda}\\
&\qquad \text{by Lemma 7.37}\\
&= \mathsf{q}^{-\sum_i(\mathsf{x}_i\otimes 1\otimes\mathsf{x}_i+1\otimes\mathsf{x}_i\otimes\mathsf{x}_i)}\mathsf{q}_3^{-2\bar\lambda}\mathcal{J}^{3,12}(\lambda)^{-1}X_V(\lambda,\mu)\mathcal{J}^{32}(\lambda)\mathsf{q}_3^{2\bar\lambda}\mathsf{q}^{\sum_i 1\otimes\mathsf{x}_i\otimes\mathsf{x}_i}\\
&= \mathsf{q}^{-\sum_i(\mathsf{x}_i\otimes 1\otimes\mathsf{x}_i+1\otimes\mathsf{x}_i\otimes\mathsf{x}_i)}\mathsf{q}_3^{-2\bar\lambda}Y(\lambda,\mu)\mathsf{q}_3^{2\bar\lambda}\mathsf{q}^{\sum_i 1\otimes\mathsf{x}_i\otimes\mathsf{x}_i} \quad \text{by Lemma 7.41,}
\end{aligned}$$

and the result follows. □

Proposition 7.43

$$X_V(\lambda,\mu) = \mathcal{J}^{3,12}(\lambda)\mathcal{J}^{12}\left(\lambda - \tfrac{1}{2}h^3\right)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2 - \tfrac{1}{2}h^3,\mu\right)\mathcal{J}^{32}(\lambda)^{-1}.$$

Proof We can write $(\mathcal{R}^{03})^{-1} = \mathsf{q}^{-\mathsf{x}_i\otimes\mathsf{x}_i} + N_1$, where N_1 consists of terms of negative weight in component 3. Now

$$\begin{aligned}
X_V(\lambda,\mu) &= \mathsf{tr}|_0\left(\Phi_\mu^{011^*}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda}(\mathsf{q}^{-\mathsf{x}_i\otimes\mathsf{x}_i}+N_1)\right)\\
&= \mathsf{tr}|_0\left(\Phi_\mu^{011^*}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda}\mathsf{q}^{-\mathsf{x}_i\otimes\mathsf{x}_i}\right) + N_2, \quad \text{where } \mathsf{wt}\, N_2 < 0 \text{ in component 3}\\
&= \mathsf{tr}|_0\left(\Phi_\mu^{011^*}\mathcal{R}^{20}\mathsf{q}_0^{2\bar\lambda}\mathsf{q}_0^{-\overline{h^3}}\right) + N_2\\
&= Z\left(\lambda - \tfrac{1}{2}h^3,\mu\right) + N_2.
\end{aligned}$$

Since $\mathcal{J}$ is lower triangular with ones on the diagonal, and since $Y(\lambda,\mu) = \mathcal{J}^{3,12}(\lambda)^{-1}X_V(\lambda,\mu)\mathcal{J}^{32}(\lambda)$, it follows that

$$Y(\lambda,\mu) = Z\left(\lambda - \tfrac{1}{2}h^3,\mu\right) + N_3, \tag{7.15}$$

where N_3 has negative weight in component 3. Now let Y_β denote the term of weight $\beta = (\beta_1,\beta_2,\beta_3)$ in Y. Extracting the term of weight β from both sides of (7.14) gives

$$\mathsf{q}_3^{2\bar\lambda}\mathsf{q}^{\sum_i(\mathsf{x}_i\otimes 1\otimes\mathsf{x}_i+1\otimes\mathsf{x}_i\otimes\mathsf{x}_i)}Y_\beta(\lambda,\mu) = Y_\beta(\lambda,\mu)\mathsf{q}_3^{2\bar\lambda}\mathsf{q}^{\sum_i 1\otimes\mathsf{x}_i\otimes\mathsf{x}_i}, \tag{7.16}$$

and if we apply both sides of (7.16) to an element v of weight $(\gamma_1,\gamma_2,\gamma_3)$, we get

$$\mathsf{q}^{\langle 2\lambda+\beta_1+\gamma_1+\beta_2+\gamma_2,\beta_3+\gamma_3\rangle} Y_\beta(\lambda,\mu)v = \mathsf{q}^{\langle 2\lambda+\gamma_2,\gamma_3\rangle} Y_\beta(\lambda,\mu)v \tag{7.17}$$

Now it is clear that unless $\beta_3 = 0$, we will have

$$\langle 2\lambda + \beta_1 + \gamma_1 + \beta_2 + \gamma_2, \beta_3 + \gamma_3\rangle \neq \langle 2\lambda + \gamma_2, \gamma_3\rangle$$

for generic λ. From this it follows that $Y_\beta = 0$ whenever $\beta_3 \neq 0$, and thus $N_3 = 0$. We thus conclude that

$$Y(\lambda,\mu) = Z\left(\lambda - \tfrac{1}{2}h^3, \mu\right),$$

and the result follows from Proposition 7.40. □

Corollary 7.44

$$\begin{aligned} &\mathsf{tr}|_0 \left(\mathcal{R}^{20}(\mathcal{R}^{03})^{-1}\Phi_\mu^{V,011^*}\mathsf{q}_0^{2\bar\lambda}\right) \\ &= \mathsf{q}_2^{-2\bar\lambda}\mathcal{J}^{3,12}(\lambda)\mathcal{J}^{12}\left(\lambda - \tfrac{1}{2}h^3\right)\Psi_V^1\left(\lambda + \tfrac{1}{2}h^2 - \tfrac{1}{2}h^3, \mu\right)\mathcal{J}^{32}(\lambda)^{-1}\mathsf{q}_2^{2\bar\lambda}. \end{aligned} \tag{7.18}$$

Proof Follows from Proposition 7.43. □

7.7 The function $\tilde{G}$

Proposition 7.45

$$\chi_W(\mathsf{q}^{2(\bar\mu+\bar\rho)})\Psi_V(\lambda,\mu) = \sum_\nu \mathsf{tr}|_{W[\nu]}\left((\tilde{G}(\lambda)\otimes \mathsf{Id})\mathbf{R}_{WV}(\lambda)\right)\Psi_V(\lambda+\nu,\mu),$$

where $\tilde{G}(\lambda) = \mathsf{q}^{-2\bar\rho}\mathbf{Q}^{-1}(\lambda+h)\mathbf{S}(\mathbf{Q})(\lambda)$

Let $\mathcal{J}(\lambda) = \sum_i c_i \otimes d_i(\lambda)$ and $\mathcal{J}^{-1}(\lambda) = \sum_i c_i' \otimes d_i'(\lambda)$. Let $\mathcal{R} = \sum_i a_i \otimes b_i$ and $\mathcal{R}^{-1} = \sum_i a_i' \otimes b_i'$. As before, u will denote the Drinfeld element (see Definition 5.9).

Lemma 7.46

$$\sum_k d_k'(\lambda)\mathbf{S}^{-1}(c_k') = \mathbf{Q}^{-1}(\lambda+h).$$

Proof The lemma follows from applying $\mathbf{m}_{321}\circ(\mathbf{S}^{-1}\otimes \mathsf{Id}\otimes\mathbf{S}^{-1})$ to the cocycle equation for $\mathbf{J}$,

$$\mathbf{J}^{12,3}(\lambda)\mathbf{J}^{12}(\lambda+h^3) = \mathbf{J}^{1,23}(\lambda)\mathbf{J}^{23}(\lambda).$$

□

Lemma 7.47

$$\sum_k d'_k(\lambda)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}(c'_k) = \mathsf{q}^{\sum_l \mathsf{x}_l^2}\mathbf{Q}^{-1}(\lambda+h)\mathbf{S}(u)\mathsf{q}^{2\bar{\lambda}}.$$

Proof From Theorem 5.8, we get

$$\mathcal{J}^{-1}(\lambda)\mathsf{q}_1^{-2\bar{\lambda}} = \mathsf{q}^{\sum_l \mathsf{x}_l^2}\mathsf{q}_1^{-2\bar{\lambda}}\mathcal{J}^{-1}(\lambda)\mathcal{R}^{21}. \tag{7.19}$$

Applying $\mathbf{m}_{21}\mathbf{S}_1$ to both sides of (7.19) gives

$$\sum_k d'_k(\lambda)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}(c'_k) = \mathsf{q}^{\sum_l \mathsf{x}_l^2}\sum_{j,k} d'_k(\lambda)a_j\mathbf{S}(b_j)\mathbf{S}(c'_k)\mathsf{q}^{2\bar{\lambda}}. \tag{7.20}$$

But $\mathbf{S}(u) = \sum_j a_j\mathbf{S}(b_j)$, so the lemma follows from Lemma 7.46. □

Lemma 7.48 *For any $z, c \in \mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$, we have*

$$\mathbf{S}(u)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}(c)\mathsf{q}^{2\bar{\rho}}z = \mathsf{q}^{2\bar{\lambda}}\mathbf{S}^{-1}(c)z\mathbf{S}(u)\mathsf{q}^{2\bar{\rho}}.$$

Proof We have $\mathbf{S}(c)\mathsf{q}^{2\bar{\rho}} = \mathsf{q}^{2\bar{\rho}}\mathbf{S}^{-1}(c)$, by Proposition 4.26. Furthermore, we know that $u\mathsf{q}^{-2\bar{\rho}}$ is central in $\widehat{\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})}$, by Proposition 5.13. Hence, $\mathbf{S}^{-1}(u\mathsf{q}^{-2\bar{\rho}}) = \mathbf{S}(\mathbf{S}^{-2}(u\mathsf{q}^{-2\bar{\rho}})) = \mathbf{S}(\mathsf{q}^{-2\bar{\rho}}u) = \mathbf{S}(u)\mathbf{S}(\mathsf{q}^{2\bar{\rho}})$ is also central, and the result follows. □

Lemma 7.49

$$\sum_i d_i^1(\lambda)\otimes \mathsf{q}^{2\bar{\lambda}}\mathbf{S}^{-1}(c_i)\mathsf{q}^{-2\bar{\lambda}}d_i^2(\lambda)$$
$$= \mathsf{q}^{\sum_l (x_l)_2((x_l)_1+(x_l)_2)}\sum_{r,s}(a'^{\,1}_r d_k^1(\lambda)\otimes \mathsf{Id})\mathbf{S}^{-1}(c_s)\mathbf{S}^{-1}(b'_r)a'^{\,2}_j d_k^2(\lambda).$$

Proof From Theorem 5.8, we get

$$\mathsf{q}_1^{2\bar{\lambda}}\mathcal{J}^{1,23}(\lambda)\mathsf{q}_1^{-2\bar{\lambda}} = (\mathcal{R}^{23,1})^{-1}\mathcal{J}^{1,23}(\lambda)\mathsf{q}^{\sum_l (x_l)_1((x_l)_2+(x_l)_3)}. \tag{7.21}$$

The lemma follows from applying $\mathbf{m}_{13}\circ\mathbf{S}_1^{-1}$ to both sides of (7.21). □

Lemma 7.50

$$\sum_i a'^{\,1}_i \otimes \mathbf{S}^{-1}(b'_i)a'^{\,2}_i = \sum_k a_k\otimes u^{-1}b_k.$$

Proof We know that $(\boldsymbol{\Delta}\otimes \mathsf{Id})(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23}$; hence,

$$(\boldsymbol{\Delta}\otimes \mathsf{Id})(\mathcal{R}^{-1}) = (\mathcal{R}^{23})^{-1}(\mathcal{R}^{13})^{-1}. \tag{7.22}$$

Applying $\mathbf{m}_{32}\circ \mathbf{S}_3^{-1}$ to both sides of (7.22) gives

$$\begin{aligned}\sum_i {a'_i}^1 \otimes \mathbf{S}^{-1}(b'_i){a'_i}^2 &= \sum_{j,k} a'_k \otimes \mathbf{S}^{-1}(b'_k)\mathbf{S}^{-1}(b'_j)a'_j \\ &= \sum_k a'_k \otimes \mathbf{S}^{-1}(b'_k)u^{-1} \quad \text{by theorem 5.11, part 3} \\ &= \sum_k a'_k \otimes u^{-1}\mathbf{S}(b'_k) \quad \text{by theorem 5.11, part 2} \\ &= u_2^{-1}(\mathsf{Id}\otimes \mathbf{S})(\mathcal{R}^{-1}) \\ &= u_2^{-1}\mathcal{R} \quad \text{by Proposition 4.41.}\end{aligned}$$

□

Lemma 7.51

$$\sum_i \mathbf{S}(c_i)d_i^1(\lambda)\otimes d_i^2(\lambda) = \mathbf{S}(\mathbf{Q})\left(\lambda+\frac{1}{2}h^2\right)_1 \mathcal{J}^{-1}\left(\lambda+\frac{1}{2}\hat{h}^1\right),$$

where $\hat{h}^1$ denotes the weight of the first component after the action of $\mathcal{J}^{-1}$. (Note that $\hat{h}^2$ is defined similarly.)

Proof Follows from applying $\mathbf{m}_{12}\circ \mathbf{S}_1$ to both sides of the equation in Lemma 7.38. □

Lemma 7.52

$$\sum_i d_i^1(\lambda)\otimes \mathsf{q}^{2\bar{\lambda}}\mathbf{S}^{-1}(c_i)\mathsf{q}^{-2\bar{\lambda}}d_i^2(\lambda)$$
$$\mathsf{q}^{\sum_l (x_l)_2((x_l)_1+(x_l)_2)}(u^{-1})_2\mathbf{S}(\mathbf{Q})(\lambda+h^1/2)_2(\mathcal{J}^{21})^{-1}(\lambda+\hat{h}^2/2)\mathcal{R}.$$

Proof Follows from Lemmas 7.49, 7.50, 7.51 and from the fact that $\mathcal{R}^{23}\mathcal{J}^{1,23} = \mathcal{J}^{1,32}\mathcal{R}^{23}$. □

Proof of Proposition 7.45 The right-hand side of (7.18) can we rewritten as

$$\mathsf{q}_2^{-2\bar{\lambda}}\left(\sum_{i,j} d_i^1(\lambda)c_j\otimes d_i^2(\lambda)d_j(\lambda-h^3/2)\otimes c_i\right)$$
$$\Psi_V(\lambda+h^2/2-h^3/2,\mu)\left(\sum_k \mathsf{Id}\otimes d'_k\otimes c'_k\right)\mathsf{q}_2^{2\bar{\lambda}}.$$

We then apply $\mathbf{m}_{23} \circ \mathbf{S}_3$, followed by $\mathsf{tr}|_{W_2}\left(\cdots \mathsf{q}_2^{2\bar{\rho}}\right)$ to both sides of (7.18). The left-hand side of (7.18) becomes

$$\begin{aligned}
&\mathsf{tr}|_{W_2}\, \mathsf{tr}|_0 \,(\mathcal{R}^{20}\mathcal{R}^{02}\Phi_\mu^{V,011^*}\mathsf{q}_0^{2\bar{\lambda}}\mathsf{q}_2^{2\bar{\rho}}) \quad \text{by Proposition 4.41}\\
&\quad = \mathsf{tr}|_0\,(C_W\Phi_\mu^{V,011^*}\mathsf{q}_0^{2\bar{\lambda}})\\
&\quad = \chi_W(\mathsf{q}^{2(\bar{\mu}+\bar{\rho})})\Psi_V(\lambda,\mu).
\end{aligned}$$

The right-hand side of (7.18) becomes

$$\begin{aligned}
&\sum_{i,j,k} d_i^1(\lambda)c_j\Psi_V\left(\lambda+h^2,\mu\right)\\
&\quad \mathsf{tr}|_{W_2}\left(\mathsf{q}^{-2\bar{\lambda}}d_i^2(\lambda)d_j\left(\lambda+\tfrac{1}{2}h^2\right)d_k'(\lambda)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}(c_k')\mathbf{S}(c_i)\mathsf{q}^{2\bar{\rho}}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\chi_W(\mathsf{q}^{2(\bar{\mu}+\bar{\rho})})\Psi_V(\lambda,\mu)\\
&\quad = \sum_{i,j,k} d_i^1(\lambda)c_j\ \mathsf{tr}|_{W_2}\left(\mathsf{q}^{-2\bar{\lambda}}d_i^2(\lambda)d_j(\lambda+h^2/2)\right)_2\Psi_V(\lambda+h^2,\mu)\\
&\qquad \times\left(d_k'(\lambda)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}(c_k')\mathbf{S}(c_i)\mathsf{q}^{2\bar{\rho}}\right)_2\\
&\quad = \sum_{i,j} d_i^1(\lambda)c_j\Psi_V(\lambda+h^2,\mu)\\
&\qquad \times\ \mathsf{tr}|_{W_2}\,\mathsf{q}^{\sum_l \mathsf{x}_l^2}\left(\mathbf{Q}^{-1}(\lambda+h^2)\mathbf{S}(u)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}(c_i)\mathsf{q}^{2\bar{\rho}-2\bar{\lambda}}d_i^2(\lambda)d_j(\lambda+h^2/2)\right)_2\\
&\qquad \text{(by Lemma 7.47 and the cyclic property of trace)}\\
&\quad = \sum_{i,j} d_i^1(\lambda)c_j\Psi_V(\lambda+h^2,\mu)\\
&\qquad \times\ \mathsf{tr}|_{W_2}\,\mathsf{q}^{\sum_l \mathsf{x}_l^2}\left(\mathbf{Q}^{-1}(\lambda+h^2)\mathsf{q}^{2\bar{\lambda}}\mathbf{S}^{-1}(c_i)\mathsf{q}^{-2\bar{\lambda}}d_i^2(\lambda)d_j(\lambda+h^2/2)\mathbf{S}(u)\mathsf{q}^{2\bar{\rho}}\right)_2\\
&\qquad \text{(by Lemma 7.48)}\\
&\quad = \sum_{\nu} \mathsf{tr}|_{W[\nu]}\left((\tilde{G}(\lambda)\otimes \mathsf{Id})\mathbf{R}_{WV}(\lambda)\right)\Psi_V(\lambda+\nu,\mu)\\
&\qquad \text{(using Lemma 7.52 and the identity } u\mathbf{S}(u)=\mathsf{q}^{-4\bar{\rho}}\text{)}.
\end{aligned}$$

□

Now let $\mathbf{J}(\lambda)=\sum_i a_i\otimes b_i(\lambda)$; then $\mathbf{Q}(\lambda)=\sum_i \mathbf{S}^{-1}(b_i)(\lambda)a_i$.

Lemma 7.53

$$\sum_i \mathbf{S}^{-1}(b_i^1(\lambda))a_i\otimes\mathbf{S}^{-1}(b_i^2(\lambda)) = \left((\mathbf{S}^{-1}\otimes\mathbf{S}^{-1})\mathbf{J}(\lambda)^{-1}\right)\left(\mathbf{Q}(\lambda-h^2)\otimes 1\right)$$

(using Sweedler's notation).

Proof $\mathbf{J}$ satisfies the dynamical twist equation:

$$\mathbf{J}^{1,23}(\lambda)\mathbf{J}^{23}(\lambda) = \mathbf{J}^{12,3}(\lambda)\mathbf{J}^{1,2}(\lambda + h^3), \quad \text{or}$$

$$\sum_{i,j} a_i \otimes b_i^1(\lambda)a_j \otimes b_i^2(\lambda)b_j(\lambda) = \sum_{k,l} a_k^1 a_l \otimes a_k^2 b_l(\lambda + h^3) \otimes b_k(\lambda). \tag{7.23}$$

Applying $(\mathbf{m}_{21} \otimes \mathsf{Id}_3)\mathbf{S}_2^{-1}$ to both sides of (7.23) gives

$$\begin{aligned}
&\sum_{i,j} \mathbf{S}^{-1}(a_j)\mathbf{S}^{-1}(b_i^1)(\lambda)a_i \otimes b_i^2(\lambda)b_j(\lambda)\\
&\quad = \sum_{k,l} \mathbf{S}^{-1}(b_l)(\lambda + h^2)\mathbf{S}^{-1}(a_k^2)a_k^1 a_l \otimes b_k(\lambda)\\
&\quad \Longrightarrow \sum_{i,j} \mathbf{S}^{-1}(a_j)\mathbf{S}^{-1}(b_i^1)(\lambda)a_i \otimes b_i^2(\lambda)b_j(\lambda)\\
&\quad = \sum_{l} \mathbf{S}^{-1}(b_l)(\lambda + h^2)a_l \otimes c(\lambda)\\
&\quad \Longrightarrow \sum_{i,j} \mathbf{S}^{-1}(a_j)\mathbf{S}^{-1}(b_i^1)(\lambda)a_i \otimes b_i^2(\lambda)b_j(\lambda)\\
&\quad = \mathbf{Q}(\lambda + h^2) \otimes 1, \quad \text{since } (\epsilon \otimes \mathsf{Id})\mathbf{J}(\lambda) = 1.
\end{aligned} \tag{7.24}$$

We then apply $\mathbf{S}_2^{-1}$ to both sides of (7.24). Since $\mathbf{S}_2(\mathbf{Q}(\lambda+h^2)\otimes 1) = \mathbf{Q}(\lambda-h^2)\otimes 1$, we get

$$\begin{aligned}
&\sum_{i,j} \mathbf{S}^{-1}(a_j)\mathbf{S}^{-1}(b_i^1(\lambda))a_i \otimes \mathbf{S}^{-1}(b_j(\lambda))\mathbf{S}^{-1}(b_i^2(\lambda)) = \mathbf{Q}(\lambda - h^2) \otimes 1\\
&\quad \Longrightarrow (\mathbf{S}^{-1} \otimes \mathbf{S}^{-1})(\mathbf{J}(\lambda)) \sum_{i} (\mathbf{S}^{-1}(b_i^1(\lambda))a_i \otimes \mathbf{S}^{-1}(b_i^2(\lambda))) = \mathbf{Q}(\lambda - h^2) \otimes 1.
\end{aligned}$$

The result follows. □

Lemma 7.54

$$\mathbf{\Delta}(\mathbf{Q}(\lambda)) = \left(\mathbf{S} \otimes \mathbf{S})\mathbf{J}^{21}(\lambda)^{-1}\right) (\mathbf{Q}(\lambda) \otimes \mathbf{Q}(\lambda - h^1))\mathbf{J}(\lambda - h^1 - h^2)^{-1}.$$

Proof First, we apply $\mathbf{\Delta}_3$ to both sides of (7.23):

$$\begin{aligned}
&\sum_{i,j} a_i \otimes b_i^1(\lambda)a_j \otimes b_i^{21}(\lambda)b_j^1(\lambda) \otimes b_i^{22}(\lambda)b_j^2(\lambda)\\
&\quad = \sum_{k,l} a_k^1 a_l \otimes a_k^2 b_l(\lambda + h^3 + h^4) \otimes b_k^1(\lambda) \otimes b_k^2(\lambda)\\
&\quad \Longrightarrow \sum_{i,j} a_i \otimes b_i^{11}(\lambda)a_j \otimes b_i^{12}(\lambda)b_j^1(\lambda) \otimes b_i^2(\lambda)b_j^2(\lambda)\\
&\quad = \sum_{k,l} a_k^1 a_l \otimes a_k^2 b_l(\lambda + h^3 + h^4) \otimes b_k^1(\lambda) \otimes b_k^2(\lambda). \qquad (7.25)
\end{aligned}$$

We then apply $(\mathbf{m}_{41} \otimes \mathbf{m}_{32}) \circ (\mathsf{Id} \otimes \mathsf{Id} \otimes \mathbf{S}^{-1} \otimes \mathbf{S}^{-1})$ to both sides of (7.25):

$$\begin{aligned}
&\sum_{i,j} \mathbf{S}^{-1}(b_j^2(\lambda))\mathbf{S}^{-1}(b_i^2(\lambda))a_i \otimes \mathbf{S}^{-1}(b_j^1(\lambda))\mathbf{S}^{-1}(b_i^{12}(\lambda))b_i^{11}(\lambda)a_j\\
&\quad = \sum_{k,l} \mathbf{S}^{-1}(b_k^2(\lambda))a_k^1 a_l \otimes \mathbf{S}^{-1}(b_k^1(\lambda))a_k^2 b_l(\lambda - h^1 - h^2)
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \sum_{i,j} \mathbf{S}^{-1}(b_j^2(\lambda))\mathbf{S}^{-1}(b_i^2(\lambda))a_i \otimes \mathbf{S}^{-1}(b_j^1(\lambda))\mathbf{S}^{-1}(b_i^{12}(\lambda))b_i^{11}(\lambda)a_j\\
&\quad = \mathbf{\Delta}(\mathbf{Q}(\lambda))\mathbf{J}(\lambda - h^1 - h^2). \qquad (7.26)
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{S}^{-1}(b_i^2) \otimes \mathbf{S}^{-1}(b_i^{12})b_i^{11} &= (\mathbf{S}^{-1} \otimes \mathbf{S}^{-1})(b_i^2 \otimes \mathbf{S}(b_i^{11})b_i^{12})\\
&= (\mathbf{S}^{-1} \otimes \mathbf{S}^{-1} \circ i \circ \epsilon)(\mathbf{\Delta}^{\mathrm{op}}(b_i))\\
&= \mathbf{S}^{-1}(b_i) \otimes 1,
\end{aligned}$$

so the left-hand side (7.26) becomes

$$\begin{aligned}
&\sum_{i,j} \mathbf{S}^{-1}(b_j^2(\lambda))\mathbf{S}^{-1}(b_i(\lambda))a_i \otimes \mathbf{S}^{-1}(b_j^1(\lambda))a_j\\
&\quad = \sum_{j} \mathbf{S}^{-1}(b_j^2(\lambda))\mathbf{Q}(\lambda) \otimes \mathbf{S}^{-1}(b_j^1(\lambda))a_j\\
&\quad = \left(\sum_{j} \mathbf{S}^{-1}(b_j^2(\lambda)) \otimes \mathbf{S}^{-1}(b_j^1(\lambda))a_j\right)(\mathbf{Q}(\lambda) \otimes 1)\\
&\quad = (\mathbf{S}^{-1} \otimes \mathbf{S}^{-1})(\mathbf{J}^{21}(\lambda))^{-1}(\mathbf{Q}(\lambda) \otimes \mathbf{Q}(\lambda - h^1)) \quad \text{by Lemma 7.53.}
\end{aligned}$$

Thus (7.26) becomes

$$\mathbf{\Delta}(\mathbf{Q}(\lambda))\mathbf{J}(\lambda - h^1 - h^2) = (\mathbf{S}^{-1} \otimes \mathbf{S}^{-1})(\mathbf{J}^{21}(\lambda))^{-1}(\mathbf{Q}(\lambda) \otimes \mathbf{Q}(\lambda - h^1)),$$

and the result follows since $\mathbf{S}^2 = \mathsf{Ad}\,\mathsf{q}^{2\bar{\rho}} \Longrightarrow (\mathbf{S}^2 \otimes \mathbf{S}^2)(\mathbf{J}(\lambda)) = \mathbf{J}(\lambda)$. □

Lemma 7.55

$$\mathbf{\Delta}(\tilde{G}(\lambda)) = \mathbf{J}(\lambda)(\tilde{G}(\lambda + h^2) \otimes \tilde{G}(\lambda))\mathbf{J}(\lambda)^{-1}.$$

Proof From Lemma 7.54, it follows that

$$\begin{aligned}&\mathbf{\Delta}(\mathbf{Q}^{-1}(\lambda + h))\\ &\quad = \mathbf{J}(\lambda)(\mathbf{Q}(\lambda + h^1 + h^2) \otimes \mathbf{Q}(\lambda + h^2))(\mathbf{S} \otimes \mathbf{S})(\mathbf{J}^{21}(\lambda + h^1 + h^2))\end{aligned}$$

and

$$\begin{aligned}&\mathbf{\Delta}(\mathbf{S}(\mathbf{Q})(\lambda))\\ &\quad = (\mathbf{S} \otimes \mathbf{S})\mathbf{J}^{21}(\lambda + h^1 + h^2)^{-1}(\mathbf{S}(\mathbf{Q})(\lambda + h^2) \otimes \mathbf{S}(\mathbf{Q})(\lambda))(\mathbf{S}^2 \otimes \mathbf{S}^2)\mathbf{J}(\lambda)^{-1}.\end{aligned}$$

Using $(\mathbf{S}^2 \otimes \mathbf{S}^2)(\mathbf{J}(\lambda)) = \mathbf{J}(\lambda)$, we conclude that

$$\begin{aligned}&\mathbf{\Delta}(\tilde{G}(\lambda))\\ &\quad = \mathbf{\Delta}(\mathsf{q}^{-2\bar{\rho}})\mathbf{\Delta}(\mathbf{Q}^{-1}(\lambda + h))\mathbf{\Delta}(\mathbf{S}(\mathbf{Q})(\lambda))\\ &\quad = (\mathsf{q}^{-2\bar{\rho}} \otimes \mathsf{q}^{-2\bar{\rho}})\mathbf{J}(\lambda)(\mathbf{Q}(\lambda + h^1 + h^2) \otimes \mathbf{Q}(\lambda + h^2))\\ &\qquad \times (\mathbf{S}(\mathbf{Q})(\lambda + h^2) \otimes \mathbf{S}(\mathbf{Q})(\lambda))\mathbf{J}(\lambda)^{-1}\\ &\quad = \mathbf{J}(\lambda)(\mathsf{q}^{-2\bar{\rho}}\mathbf{Q}(\lambda + h^1 + h^2)\mathbf{S}(\mathbf{Q})(\lambda + h^2) \otimes \mathsf{q}^{-2\bar{\rho}}\mathbf{Q}(\lambda + h^2)\mathbf{S}(\mathbf{Q})(\lambda))\mathbf{J}(\lambda)^{-1}\\ &\quad = \mathbf{J}(\lambda)(\tilde{G}(\lambda + h^2) \otimes \tilde{G}(\lambda))\mathbf{J}(\lambda)^{-1}.\end{aligned}$$

□

Lemma 7.56 *Suppose that we have functions*

$$\eta_i : \mathfrak{h}^* \to \mathfrak{U}(\mathfrak{g})[[\hbar]], \quad i = 1, \ldots, \mathsf{dim}\,\mathfrak{h},$$

satisfying the "zero curvature equations",

$$\eta_i(\lambda + \hbar\omega_j)\eta_j(\lambda) = \eta_j(\lambda + \hbar\omega_i)\eta_i(\lambda) \quad \textit{for all } i, j, \tag{7.27}$$

with $\eta_i(0) = 1$ *for all* i*. Then there exists* $g : \mathfrak{h}^* \to \mathfrak{U}(\mathfrak{g})[[\hbar]]$ *such that*

$$g(\lambda + \hbar\omega_j) = \eta_j(\lambda)g(\lambda) \quad \textit{for all } j. \tag{7.28}$$

Proof We will prove by induction that for each $N \geq 1$, there exists $g^{(N)} : \mathfrak{h}^* \to \mathfrak{U}(\mathfrak{g})[[\hbar]]$ such that

$$g^{(N)}(\lambda + \hbar\omega_j) \equiv \eta_j(\lambda)g^{(N)}(\lambda) \mod \hbar^N \quad \text{for all } j, \tag{7.29}$$

with the condition that

$$g^{(N+1)} \equiv g^{(N)} \mod \hbar^{N-1} \quad \text{for all } N. \tag{7.30}$$

The result will then follow upon taking $g = \lim_{N\to\infty} g^{(N)}$.

Since $\eta_i(0) = 1$ for all i, we can take $g^{(0)} = 1$. Given $g^{(N)}$ satisfying (7.29), we let

$$\eta_i^{(N)}(\lambda) = \frac{g^{(N)}(\lambda + \hbar\omega_i)}{g^{(N)}(\lambda)} \quad \text{for all } \lambda, i,$$

and

$$\gamma_i^{(N)}(\lambda) = \frac{\eta_i(\lambda)}{\eta_i^{(N)}(\lambda)} \quad \text{for all } \lambda, i.$$

It is clear that for all i,

$$\gamma_i^{(N)}(\lambda) = 1 + O(\hbar^N) \text{ for all } \lambda \text{ and } \gamma_i^{(N)} \text{satisfies (7.27).}$$

Writing $\gamma_i^{(N)}(\lambda) = 1 + \hbar^N b_i(\lambda) + O(\hbar^{N+1})$, and extracting the $\hbar^{N+1}$ term of the zero curvature equation (7.27) for $\gamma_i^{(N)}$, we get

$$\frac{\partial b_i}{\partial \omega_j} - \frac{\partial b_j}{\partial \omega_i} = 0 \quad \text{for all } i, j;$$

hence, there (locally) exists $\phi : \mathfrak{h}^* \to \mathfrak{U}(\mathfrak{g})$ such that $b_i = \partial\phi/\partial\omega_i$ for all i. We then take

$$g^{(N+1)}(\lambda) = g^{(N)}(\lambda)(1 + \hbar^{N-1}\phi(\lambda));$$

we see that

$$\begin{aligned}
&\eta_i(\lambda)g^{(N+1)}(\lambda) \\
&\quad = \gamma_i^{(N)}(\lambda)\eta_i^{(N)}(\lambda)g^{(N)}(\lambda)(1 + \hbar^{N-1}\phi(\lambda)) \\
&\quad = \gamma_i^{(N)}(\lambda)g^{(N)}(\lambda + \hbar\omega_i)(1 + \hbar^{N-1}\phi(\lambda)) \\
&\quad = (1 + \hbar^N b_i(\lambda) + O(\hbar^{N+1}))g^{(N+1)}(\lambda + \hbar\omega_i)\frac{1 + \hbar^{N-1}\phi(\lambda)}{1 + \hbar^{N-1}\phi(\lambda + \hbar\omega_i)}. \qquad (7.31)
\end{aligned}$$

But

$$\begin{aligned}
\frac{1 + \hbar^{N-1}\phi(\lambda + \hbar\omega_i)}{1 + \hbar^{N-1}\phi(\lambda)} &= \frac{1 + \hbar^{N-1}\phi(\lambda) + \hbar^N \frac{\partial\phi}{\partial\omega_i} + O(\hbar^{N+1})}{1 + \hbar^{N-1}\phi(\lambda)} \\
&= 1 + \hbar^N b_i(\lambda) + O(\hbar^{N+1}).
\end{aligned}$$

Thus (7.31) becomes

$$\eta_i(\lambda)g^{(N+1)}(\lambda) = (1 + O(\hbar^{N+1}))g^{(N+1)}(\lambda + \hbar\omega_i),$$

so $g^{(N+1)}$ satisfies (7.29). It is clear that the condition (7.30) is satisfied, so the proof is complete. □

Now consider the equation

$$\mathbf{\Delta}(X(\lambda,\hbar)) = \mathbf{J}\left(\frac{\lambda}{\hbar}\right)\left(X(\lambda+\hbar h^2,\hbar)\otimes X(\lambda,\hbar)\right)\mathbf{J}^{-1}\left(\frac{\lambda}{\hbar}\right), \tag{7.32}$$

one of whose solutions is $\tilde{G}\left(\frac{\lambda}{\hbar}\right)$, which is a Taylor series in $\hbar$ with constant term 1. It turns out that this allows one to find $\tilde{G}\left(\frac{\lambda}{\hbar}\right)$ very explicitly. Namely, we have the following proposition.

Proposition 7.57 *Let $X(\lambda,\hbar)$ be a solution of (7.32) of zero weight and constant term 1. Then,*

1. *If $X(\lambda,\hbar)$ acts by 1 on the highest weight vectors of fundamental representations, then $X(\lambda,\hbar)=1$.*
2. *In general, $X(\lambda,\hbar)=\frac{g(\lambda+\hbar h)}{g(\lambda)}$, for some $g:\mathfrak{h}^*\to\mathfrak{U}(\mathfrak{g})$.*

Proof For 1, fix $\lambda=\lambda_0$. Suppose that $X(\lambda_0,\hbar)\neq 1$ and let m be the lowest nonzero power of $\hbar$ in $X(\lambda_0,\hbar)-1$. Let $\bar{X}_0=\lim_{\hbar\to 0}X(\lambda_0,\hbar)/\hbar^m\in\mathfrak{U}(\mathfrak{g})$. Then $\bar{X}_0\neq 0$, and (7.32) becomes

$$\begin{aligned}&\mathbf{\Delta}(1+\hbar^m\bar{X}_0+O(\hbar^{m+1}))\\ &\quad=\mathbf{J}\left(\frac{\lambda}{\hbar}\right)\left(1\otimes 1+\hbar^m\bar{X}_0\otimes 1+1\otimes\hbar^m\bar{X}_0+O(\hbar^{m+1})\right)\mathbf{J}^{-1}\left(\frac{\lambda}{\hbar}\right).\end{aligned} \tag{7.33}$$

Extracting the coefficient of $\hbar^m$ from both sides of (7.33) gives $\mathbf{\Delta}(\bar{X}_0)=\bar{X}_0\otimes 1+1\otimes\bar{X}_0$, which implies that $\bar{X}_0\in\mathfrak{g}$. But $\bar{X}_0$ has zero weight, so $\bar{X}_0\in\mathfrak{h}$. We also have $\bar{X}_0\mathbf{v}_{\omega_i}=0$ and hence $\omega_i(\bar{X}_0)=0$ for all i; thus $\bar{X}_0=0$, which is a contradiction. Therefore, $X(\lambda_0,\hbar)=1$, and the proof of 1 is complete.

Let us now prove 2. For all i, let $X(\lambda,\hbar)\mathbf{v}_{\omega_i}=\eta_i(\lambda)\mathbf{v}_{\omega_i}$. We will consider the action of $X(\lambda,\hbar)$ on $\mathbf{v}_{\omega_i+\omega_j}\in V_{\omega_i+\omega_j}$, which is the same as $\mathbf{v}_{\omega_i}\otimes\mathbf{v}_{\omega_j}\in V_{\omega_i}\otimes V_{\omega_j}$. (Note that $V_{\omega_i+\omega_j}\subset V_{\omega_i}\otimes V_{\omega_j}$.)

Since $\mathbf{J}$ is upper triangular with ones on the diagonal, it must act as 1 on highest weight vectors. Thus, for all i,j,

$$\begin{aligned}X(\lambda,\hbar)\mathbf{v}_{\omega_i+\omega_j}&=\mathbf{\Delta}(X(\lambda,\hbar))(\mathbf{v}_{\omega_i}\otimes\mathbf{v}_{\omega_j})\\ &=\eta_i(\lambda+\hbar\omega_j)\eta_j(\lambda)\mathbf{v}_{\omega_i+\omega_j}.\end{aligned}$$

Then the η_i satisfy (7.27), so Lemma 7.56 implies the existence of a solution $g(\lambda)$ of (7.28). Taking $\tilde{X}(\lambda,\hbar)=X(\lambda,\hbar)g(\lambda)/g(\lambda+\hbar h)$, we have $\tilde{X}(\lambda,\hbar)\mathbf{v}_{\omega_i}=\mathbf{v}_{\omega_i}$ for all i; by part 1, we get $\tilde{X}(\lambda,\hbar)=1$, and hence $X(\lambda,\hbar)=g(\lambda+\hbar h)/g(\lambda)$. This proves 2. □

Corollary 7.58

$$\tilde{G}(\lambda)=\frac{f(\lambda+h)}{f(\lambda)}\quad\textit{for some } f.$$

Here h is a simplification of the dynamical notation h^1.

Proof Part 2 of Proposition 7.57 implies that $\tilde{G}(\lambda/\hbar) = g(\lambda + \hbar h)/g(\lambda)$; thus, $\tilde{G}(\lambda) = f(\lambda + h)/f(\lambda)$, where $f(x) = g(\hbar x)$. □

We showed before that $\tilde{G}(\lambda) = f(\lambda + h)/f(\lambda)$ for some f. It remains to determine $f(\lambda)$.

Proposition 7.59 *$f(\lambda) = \delta_{\mathsf{q}}(\lambda)$, where $\delta_{\mathsf{q}}(\lambda) = \mathsf{q}^{2\langle\lambda,\rho\rangle} \prod_{\alpha\in\mathsf{R}_+}(1 - \mathsf{q}^{-2\langle\lambda,\alpha\rangle})$.*

Proof Let $\mathcal{M}_W = \sum_\nu \mathsf{tr}|_{W[\nu]} \left(\left(\tilde{G}(\lambda) \otimes \mathsf{Id}\right) \mathbf{R}_{WV}(\lambda)\right) T_\nu$. Recall that

$$\mathcal{M}_W F_V = \chi_W(\mathsf{q}^{-2\bar{\lambda}}) F_V,$$

and hence

$$\mathcal{M}_W \Psi_V(\lambda, \mu) = \chi_W\left(\mathsf{q}^{-2(\bar{\mu}+\bar{\rho})}\right) \Psi_V(\lambda, \mu), \tag{7.34}$$

is satisfied for all representations V. Let us take $V = \mathbf{C}$. Then $M_\mu \otimes \mathbf{C} \cong M_\mu$, so the intertwiners are scalar; hence,

$$\begin{aligned}
\Psi_V(\lambda, \mu) &= \mathsf{tr}|_{M_\mu} \mathsf{q}^{2\bar{\lambda}} \\
&= \sum_{\beta\geq 0} \mathsf{q}^{2\langle\lambda,\mu-\beta\rangle} \dim M_\mu[\mu - \beta] \\
&= \frac{\mathsf{q}^{2\langle\lambda,\mu\rangle}}{\prod_{\alpha\in\mathsf{R}_+}\left(1 - \mathsf{q}^{-2\langle\lambda,\alpha\rangle}\right)} \\
&= \frac{\mathsf{q}^{2\langle\lambda,\mu+\rho\rangle}}{\delta_{\mathsf{q}}(\lambda)}.
\end{aligned} \tag{7.35}$$

Applying $\mathcal{M}_W$ to both sides of (7.35), we get

$$\mathcal{M}_W \Psi_V(\lambda, \mu) = \sum_\nu \dim W[\nu] \frac{f(\lambda + \nu)}{f(\lambda)} T_\nu \frac{\mathsf{q}^{2\langle\lambda,\mu+\rho\rangle}}{\delta_{\mathsf{q}}(\lambda)}. \tag{7.36}$$

From (7.34) and (7.36), we get

$$\begin{aligned}
\sum_\nu \dim W[\nu] \frac{f(\lambda + \nu)}{f(\lambda)} T_\nu \frac{\mathsf{q}^{2\langle\lambda,\mu+\rho\rangle}}{\delta_{\mathsf{q}}(\lambda)} &= \chi_W\left(\mathsf{q}^{2(\mu+\rho)}\right) \frac{\mathsf{q}^{2\langle\lambda,\mu+\rho\rangle}}{\delta_{\mathsf{q}}(\lambda)} \\
\sum_\nu \dim W[\nu] \frac{f(\lambda + \nu)}{f(\lambda)} \frac{\mathsf{q}^{2\langle\lambda+\nu,\mu+\rho\rangle}}{\delta_{\mathsf{q}}(\lambda + \nu)} &= \chi_W\left(\mathsf{q}^{2(\mu+\rho)}\right) \frac{\mathsf{q}^{2\langle\lambda,\mu+\rho\rangle}}{\delta_{\mathsf{q}}(\lambda)}.
\end{aligned} \tag{7.37}$$

Clearly, $f(\lambda) = \delta_{\mathsf{q}}(\lambda)$ satisfies (7.37); but one can easily check that a rational solution to (7.37) is unique up to scaling. This completes the proof. □

7.8 Macdonald–Ruijsenaars equations

Now we can finally prove the first statement of Theorem 7.7:

$$\mathbf{D}_W^{(\lambda)} F_V(\lambda,\mu) = \chi_W(\mathsf{q}^{-2\bar{\mu}}) F_V(\lambda,\mu).$$

Proof We have

$$\begin{aligned}\chi_W(\mathsf{q}^{2(\overline{\mu+\rho})})\Psi_V(\lambda,\mu) &= \sum_\nu \mathsf{tr}|_{W[\nu]}\left(\frac{\delta_\mathsf{q}(\lambda+\nu)}{\delta_\mathsf{q}(\lambda)}\otimes \mathsf{Id}\right)\mathbf{R}_{WV}(\lambda)\Psi_V(\lambda+\nu,\mu)\\ &= \delta_\mathsf{q}(\lambda)^{-1}\sum_\nu \mathsf{tr}|_{W[\nu]}\,\mathbf{R}_{WV}(\lambda)T_\nu\delta_\mathsf{q}(\lambda)\Psi_V(\lambda,\mu)\\ &= \delta_\mathsf{q}(\lambda)^{-1}\mathbf{D}_W^{(\lambda)}\delta_\mathsf{q}(\lambda)\Psi(\lambda,\mu).\end{aligned}$$

Hence

$$\chi_W(\mathsf{q}^{2(\overline{\mu+\rho})})\Psi_V(\lambda,\mu)\delta_\mathsf{q}(\lambda) = \mathbf{D}_W^{(\lambda)}\Psi(\lambda,\mu)\delta_\mathsf{q}(\lambda).$$

The result follows upon first replacing μ by $-(\mu+\rho)$ and then right-multiplying both sides by $\mathbf{Q}^{-1}(\mu)$. □

7.9 Dual Macdonald–Ruijsenaars equations

We consider $F_V(\lambda,\mu)\in V[0]\otimes V[0]^*$. Let $\{v_i\},\{v_i^*\}$ be dual bases for V, V^*. Let

$$\Phi_\mu^V : M_\mu \to \oplus_\beta M_{\mu+\beta}\otimes V\otimes V^*$$

be defined by

$$\Phi_\mu^V \stackrel{\text{def}}{=} \sum_i \Phi_\mu^{v_i}\otimes v_i^*.$$

For generic μ, we have $M_\mu\otimes W \cong \sum_\nu W[\nu]\otimes M_{\mu+\nu}$. Let

$$\eta : \sum_\nu W[\nu]\otimes M_{\mu+\nu} \to M_\mu\otimes W$$

be defined by

$$\eta(w\otimes y) \stackrel{\text{def}}{=} \Phi_\mu^w(y),$$

and

$$E^\nu : W[\nu]\otimes M_{\mu+\nu}\to \oplus_\beta W[\nu]\otimes M_{\mu+\beta}\otimes V\otimes V^*$$

be defined by

$$E^\nu \stackrel{\text{def}}{=} \eta^{-1}P_{V\otimes V^*,W}\mathcal{R}_{VW}\left(\Phi_\mu^V\otimes \mathsf{Id}_W\right)\eta.$$

By the defining properties of intertwiners, we can write

$$E^\nu = L_{WV^*}(\mu,\nu)\left(\mathsf{Id}_{W[\nu]}\otimes\Phi_{\mu+\nu}^V\right),$$

where

$$L_{WV^*}(\mu,\nu) : W[\nu]\otimes V^*\to W\otimes V^*$$

is linear.

Lemma 7.60

$$L_{WV^*}(\mu,\nu) = R_{WV}(\mu+\nu)^{t_2},$$

where t_2 is the adjoint in the "V" component.

Proof We have

$$P_{V\otimes V^*,W}\mathcal{R}_{VW}(\Phi_\mu^V \otimes \mathsf{Id}_W)\eta = \eta L_{WV^*}(\mu)(\mathsf{Id}_W \otimes \Phi_{\mu+\nu}^V), \tag{7.38}$$

where both sides are defined on $\sum_\nu W[\nu] \otimes M_{\mu+\nu}$. Let us apply both sides of (7.38) to $w \otimes y$, where $w \in W[\nu]$ and $y \in M_{\mu+\nu}$.

For the left-hand side, we get

$$\begin{aligned} &P_{V\otimes V^*,W}\mathcal{R}_{VW}(\Phi_\mu^V \otimes \mathsf{Id}_W)\eta(w\otimes y) \\ &\quad = \sum_i P_{V\otimes V^*,W}\mathcal{R}_{VW}(\Phi_\mu^{v_i} \otimes v_i^* \otimes \mathsf{Id}_W)\Phi_{\mu+\nu}^W y \\ &\quad = \sum_i \left(P_{VW}\mathcal{R}_{VW}(\Phi_\mu^{v_i} \otimes \mathsf{Id}_W)\Phi_{\mu+\nu}^W y\right) \otimes v_i^*. \end{aligned} \tag{7.39}$$

Now let $L_{WV^*}(\mu) = \sum_j p_j \otimes q_j$. Then the right-hand side of (7.38) becomes

$$\begin{aligned} &\eta L_{WV^*}(\mu)(\mathsf{Id}_W \otimes \Phi_{\mu+\nu}^V)(w\otimes y) \\ &\quad = \eta L_{WV^*}(\mu)\left(\sum_i w \otimes \Phi_{\mu+\nu}^{v_i} y \otimes v_j^*\right) \\ &\quad = \sum_{i,j} \eta\left(p_j w \otimes \Phi_{\mu+\nu}^{v_i} y \otimes q_j v_i^*\right) \\ &\quad = \sum_{i,j} \left(\Phi_{\mu+\nu-\mathsf{wt}\, v_i}^{p_j w} \otimes \mathsf{Id}\right)\left(\Phi_{\mu+\nu}^{v_i} y \otimes q_j v_i^*\right) \\ &\quad = \sum_{i,j} \left(\Phi_{\mu+\nu-\mathsf{wt}\, v_i}^{p_j w} \otimes \mathsf{Id}\right)\left(\Phi_{\mu+\nu}^{q_j^* v_i} y \otimes v_i^*\right). \end{aligned} \tag{7.40}$$

Comparing (7.39) and (7.40), we get that for all homogeneous $v \in V$,

$$P_{VW}\mathcal{R}_{VW}(\Phi_\mu^v \otimes \mathsf{Id})\Phi_{\mu+\nu}^W y = \sum_j \left(\Phi_{\mu+\nu-\mathsf{wt}\, v}^{p_j w} \otimes \mathsf{Id}\right)\Phi_{\mu+\nu}^{q_j^* v} y. \tag{7.41}$$

Taking expectation values in (7.41), we get

$$\begin{aligned}
&P_{VW}\mathcal{R}_{VW}J_{VW}(\mu+\nu)(v\otimes w)\\
&\quad = J_{WV}(\mu+\nu)\left(\sum_j p_j\otimes q_j^*\right)P_{VW}(v\otimes w)\\
&\quad \Longrightarrow \sum_j p_j\otimes q_j^* = J_{WV}^{-1}(\mu+\nu)\mathcal{R}^{21}J_{WV}^{21}(\mu+\nu)\\
&\quad \Longrightarrow \sum_j p_j\otimes q_j^* = R_{WV}(\mu+\nu)\\
&\quad \Longrightarrow L_{WV^*}(\mu+\nu) = R_{WV}(\mu+\nu)^{t_2},
\end{aligned}$$

which proves the lemma. □

Lemma 7.61

$$\begin{aligned}
&\mathbf{R}_{WV}(\mu)^{t_1 t_2}\\
&\quad = \left(\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1)\right)\mathbf{R}_{W^*V^*}(\mu-h^1-h^2)\left(\mathbf{Q}^{-1}(\mu-h^2)\otimes\mathbf{Q}^{-1}(\mu)\right),
\end{aligned} \tag{7.42}$$

where t_1 and t_2 are the adjoints in the "W" and "V" components, respectively.

Proof Note that

$$\left(\mathbf{R}|_{W\otimes V}\right)^{t_1 t_2} = (\mathbf{S}\otimes\mathbf{S})(\mathbf{R})(\mu)|_{W^*\otimes V^*},$$

so it is enough to show that

$$(\mathbf{S}\otimes\mathbf{S})(\mathbf{R})(\mu) = (\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1))\mathbf{R}(\mu-h^1-h^2)(\mathbf{Q}^{-1}(\mu-h^2)\otimes\mathbf{Q}^{-1}(\mu)).$$

We know that

$$\mathbf{R}(\mu) = \mathbf{J}^{-1}(\mu)\mathcal{R}^{21}\mathbf{J}^{21}(\mu). \tag{7.43}$$

Now recall that

$$\boldsymbol{\Delta}(\mathbf{Q}(\mu)) = (\mathbf{S}\otimes\mathbf{S})(\mathbf{J}^{21}(\mu)^{-1})(\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1))\mathbf{J}(\mu-h^1-h^2)^{-1};$$

hence,

$$(\mathbf{S}\otimes\mathbf{S})(\mathbf{J}^{21}(\mu)) = (\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1))\mathbf{J}(\mu-h^1-h^2)^{-1}\boldsymbol{\Delta}(\mathbf{Q}^{-1}(\mu)) \tag{7.44}$$

and

$$(\mathbf{S}\otimes\mathbf{S})(\mathbf{J}^{-1}(\mu)) = \boldsymbol{\Delta}^{21}(\mathbf{Q}(\mu))\mathbf{J}^{21}(\mu-h^1-h^2)(\mathbf{Q}^{-1}(\mu-h^2)\otimes\mathbf{Q}^{-1}(\mu)); \tag{7.45}$$

also, recall

$$(\mathbf{S}\otimes\mathbf{S})(\mathcal{R}) = \mathcal{R}. \tag{7.46}$$

Using (7.43), (7.44), (7.45) and (7.46), we get

$$
\begin{aligned}
&(\mathbf{S}\otimes\mathbf{S})(\mathbf{R}(\mu))\\
&\quad=(\mathbf{S}\otimes\mathbf{S})(\mathbf{J}^{21}(\mu))(\mathbf{S}\otimes\mathbf{S})(\mathcal{R})(\mathbf{S}\otimes\mathbf{S})(\mathbf{J}^{-1}(\mu))\\
&\quad=(\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1))\mathbf{J}(\mu-h^1-h^2)^{-1}\boldsymbol{\Delta}(\mathbf{Q}^{-1}(\mu))\mathcal{R}^{21}\boldsymbol{\Delta}^{21}(\mathbf{Q}(\mu))\\
&\qquad\mathbf{J}^{21}(\mu-h^1-h^2)(\mathbf{Q}^{-1}(\mu-h^2)\otimes\mathbf{Q}^{-1}(\mu))\\
&\quad=(\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1))\mathbf{J}(\mu-h^1-h^2)^{-1}\mathcal{R}^{21}\mathbf{J}^{21}(\mu-h^1-h^2)\\
&\qquad(\mathbf{Q}^{-1}(\mu-h^2)\otimes\mathbf{Q}^{-1}(\mu))\quad\text{using }\mathcal{R}^{21}\boldsymbol{\Delta}^{21}(x)=\boldsymbol{\Delta}(x)\mathcal{R}^{21},x=\mathbf{Q}(\mu)\\
&\quad=(\mathbf{Q}(\mu)\otimes\mathbf{Q}(\mu-h^1))\mathbf{R}(\mu-h^1-h^2)(\mathbf{Q}^{-1}(\mu-h^2)\otimes\mathbf{Q}^{-1}(\mu)),
\end{aligned}
$$

and this completes the proof. □

Lemma 7.62 *The trace of*

$$\left(\Phi_\mu^V\otimes\mathsf{Id}_W\right)\left(\mathsf{q}^{2\bar\lambda}\otimes\mathsf{q}^{2\bar\lambda}\right):M_\mu\otimes W\to M_\mu\otimes V\otimes V^*\otimes W.$$

is equal to $\Psi_V(\lambda,\mu)\chi_W(\mathsf{q}^{2\bar\lambda})$

Proof We use the fact that the trace on a tensor product is the product of the traces on each component:

$$
\begin{aligned}
\mathsf{tr}|_{M_\mu\otimes W}\left(\Phi_\mu^V\otimes\mathsf{Id}_W\right)\left(\mathsf{q}^{2\bar\lambda}\otimes\mathsf{q}^{2\bar\lambda}\right)&=\left(\mathsf{tr}|_{M_\mu}\Phi_\mu^V\mathsf{q}^{2\bar\lambda}\right)\left(\mathsf{tr}|_W\,\mathsf{q}^{2\bar\lambda}\right)\\
&=\Psi_V(\lambda,\mu)\chi_W(\mathsf{q}^{2\bar\lambda}).
\end{aligned}
$$

□

Now we are ready to prove the second statement of Theorem 7.7:

$$\mathbf{D}_W^{(\mu,V^*)}F_V(\lambda,\mu)=\chi_W(\mathsf{q}^{-2\bar\lambda})F_V(\lambda,\mu).$$

Proof By Lemma 7.60, we have

$$\eta^{-1}P_{V\otimes V^*,W}\mathcal{R}_{VW}\left(\Phi_\mu^V\otimes\mathsf{Id}_W\right)\eta=E^\nu=R_{WV}(\mu+\nu)^{t_2}(\mathsf{Id}_{W[\nu]}\otimes\Phi_{\mu+\nu}^V);\tag{7.47}$$

we then multiply both sides of (7.47) on the right by $\mathsf{q}^{2\bar\lambda}$ and take the trace. For the left-hand side, we get

$$\mathsf{tr}\left(P_{V\otimes V^*,W}\mathcal{R}_{VW}\left(\Phi_\mu^V\otimes\mathsf{Id}_W\right)\left(\mathsf{q}^{2\bar\lambda}\Big|_{M_\mu}\otimes\mathsf{q}^{2\bar\lambda}\Big|_W\right)\right).\tag{7.48}$$

Now

$$\mathsf{tr}|_{M_\mu}\left(\mathcal{R}_{WV}\Phi_\mu^V\,\mathsf{q}^{2\bar\lambda}\Big|_{M_\mu}\right)=\mathsf{tr}|_{M_\mu}\left(\Phi_\mu^V\,\mathsf{q}^{2\bar\lambda}\Big|_{M_\mu}\right),$$

since $\Phi^V_\mu(M_\mu) \subset M_\mu \otimes V[0] \otimes V^*[0]$. Also,

$$\mathsf{tr}|_W \, \mathcal{R}_{VW} \, \mathsf{q}^{2\bar\lambda}\Big|_W = \mathsf{tr}|_W \, \mathsf{q}^{\sum_i \mathsf{x}_i|_V \otimes \mathsf{x}_i|_W} \, \mathsf{q}^{2\bar\lambda}\Big|_W = \mathsf{tr}|_W \, \mathsf{q}^{2\bar\lambda},$$

since we are in $V[0]$. This shows that we can omit the $\mathcal{R}_{WV}$ from (7.48)—it does not contribute to the trace. Thus (7.48) equals

$$\begin{aligned} &\mathsf{tr}\left((\Phi^V_\mu \otimes \mathsf{Id}_W) \left(\mathsf{q}^{2\bar\lambda}\Big|_{M_\mu} \otimes \mathsf{q}^{2\bar\lambda}\Big|_W \right)\right) \\ &= \Psi_V(\lambda,\mu)\chi_W(\mathsf{q}^{2\bar\lambda}) \quad \text{using Lemma 7.62} \\ &= \Psi_V(\lambda,\mu)\chi_{W^*}(\mathsf{q}^{-2\bar\lambda}). \end{aligned} \tag{7.49}$$

For the right-hand side of (7.47), we get

$$\begin{aligned} &\sum_\nu \mathsf{tr}\, R_{WV}(\mu+\nu)^{t_2}(\mathsf{Id}_{W[\nu]} \otimes \Phi^V_{\mu+\nu})\mathsf{q}^{2\bar\lambda} \\ &\quad = \sum_\nu \left(\mathsf{tr}|_{W[\nu]} \, R_{WV}(\mu+\nu)^{t_2} \right) \Psi_V(\lambda, \mu+\nu) \\ &\quad = \sum_\nu \left(\mathsf{tr}|_{W^*[-\nu]} \, R_{WV}(\mu+\nu)^{t_1 t_2} \right) \Psi_V(\lambda, \mu+\nu) \quad \text{(pass to the dual)}. \end{aligned} \tag{7.50}$$

Equating (7.49) and (7.50), and making the substitution $\mu \to -\mu - \rho$, we obtain

$$\chi_{W^*}(\mathsf{q}^{-2\bar\lambda})\Psi_V(\lambda, -\mu-\rho) = \sum_\nu \mathsf{tr}|_{W^*[-\nu]} \, \mathbf{R}_{WV}(\mu-\nu)^{t_1t_2}\Psi_V(\lambda, -\mu-\rho+\nu). \tag{7.51}$$

Now let $\varphi(\lambda,\mu) = \Psi(\lambda, -\mu-\rho)\delta_{\mathsf{q}}(\lambda)$, so that $F_V(\lambda,\mu) = \mathbf{Q}^{-1}(\mu)\varphi(\lambda,\mu)$. Then, substituting $\nu \to -\nu$ in (7.51) and multiplying both sides of (7.51) by $\delta_{\mathsf{q}}(\lambda)$ gives

$$\begin{aligned} &\varphi_V(\lambda,\mu)\chi_{W^*}(\mathsf{q}^{-2\bar\lambda}) \\ &\quad = \sum_\nu \mathsf{tr}|_{W^*[\nu]} \, \mathbf{R}_{WV}(\mu+\nu)^{t_1t_2}\varphi_V(\lambda,\mu+\nu) \\ &\quad \Longrightarrow \mathbf{Q}^{-1}(\mu)\varphi_V(\lambda,\mu)\chi_{W^*}(\mathsf{q}^{-2\bar\lambda}) \\ &\quad = \mathbf{Q}^{-1}(\mu)\sum_\nu \mathsf{tr}|_{W^*[\nu]} \, \mathbf{R}_{WV}(\mu+\nu)^{t_1t_2}\mathbf{Q}(\mu+\nu)\mathbf{Q}^{-1}(\mu+\nu)\varphi_V(\lambda,\mu+\nu) \\ &\quad \Longrightarrow F_V(\lambda,\mu)\chi_{W^*}(\mathsf{q}^{-2\bar\lambda}) \\ &\quad = \sum_\nu \mathsf{tr}|_{W^*[\nu]} \, (\mathsf{Id} \otimes \mathbf{Q}^{-1}(\mu))\mathbf{R}_{WV}(\mu+\nu)^{t_1t_2}(\mathsf{Id} \otimes \mathbf{Q}(\mu+\nu))F_V(\lambda,\mu+\nu). \end{aligned} \tag{7.52}$$

We will now use Lemma 7.61. Making the substitution $\mu \to \mu + h^1 + h^2$ in (7.42), we get

$$\begin{aligned}&\mathbf{R}_{WV}(\mu+h^1+h^2)^{t_1t_2}\\ &\quad=\left(\mathbf{Q}(\mu+h^1+h^2)\otimes\mathbf{Q}(\mu+h^2)\right)\mathbf{R}_{W^*V^*}(\mu)\\ &\quad\quad(\mathbf{Q}^{-1}(\mu+h^1)\otimes\mathbf{Q}^{-1}(\mu+h^1+h^2)).\end{aligned}\tag{7.53}$$

Substituting (7.53) into (7.52) gives the following (note that $\mathbf{R}_{WV}(\mu+\nu)^{t_1t_2}$ acts on $W^*[\nu]\otimes V^*[0]$ in (7.52)):

$$\begin{aligned}&\chi_{W^*}(\mathsf{q}^{-2\bar\lambda})F_V(\lambda,\mu)\\ &\quad=\sum_\nu \mathsf{tr}|_{W^*[\nu]}(\mathsf{Id}\otimes\mathbf{Q}^{-1}(\mu))(\mathbf{Q}(\mu+\nu)\otimes\mathbf{Q}(\mu))\mathbf{R}_{W^*V^*}(\mu)\\ &\quad\quad(\mathbf{Q}^{-1}(\mu+\nu)\otimes\mathbf{Q}^{-1}(\mu+\nu))(\mathsf{Id}\otimes\mathbf{Q}(\mu+\nu))F_V(\lambda,\mu+\nu)\\ &\quad=\sum_\nu \mathsf{tr}|_{W^*[\nu]}(\mathbf{Q}(\mu+\nu)\otimes\mathsf{Id})\mathbf{R}_{W^*V^*}(\mu)(\mathbf{Q}^{-1}(\mu+\nu)\otimes\mathsf{Id})F_V(\lambda,\mu+\nu)\\ &\quad=\sum_\nu \mathsf{tr}|_{W^*[\nu]}\mathbf{R}_{W^*V^*}(\mu)F_V(\lambda,\mu+\nu)\\ &\quad=\mathbf{D}_W^{(\mu,V^*)}F_V(\lambda,\mu).\end{aligned}$$

This proves the theorem. □

7.10 The symmetry identity

Lemma 7.63 (Etingof and Styrkas (1998)) *The rational functions representing the matrix elements of the operator Φ_μ^v admit Taylor expansions in*

$$\mathbf{C}[[\mathsf{q}^{\langle\alpha_1,\mu\rangle},\dots,\mathsf{q}^{\langle\alpha_r,\mu\rangle}]].$$

Sketch of proof It suffices to show that the coefficients of the vector $w\stackrel{\text{def}}{=}\Phi_\mu^v\mathbf{v}_\mu$ have this property. This follows from the recursive equations on these coefficients coming from the equations $\mathsf{E}_iw=0$. □

Corollary 7.64 *For fixed λ, we have*

$$F_V(\lambda,\mu)\in\mathsf{q}^{-2\langle\lambda,\mu\rangle}\mathbf{C}[[\mathsf{q}^{-\langle\alpha_1,\mu\rangle},\dots,\mathsf{q}^{-\langle\alpha_r,\mu\rangle}]]\otimes\mathsf{End}\,V[0].$$

We now prove the third statement of Theorem 7.7:

$$F_V(\lambda,\mu)=F_{V^*}(\mu,\lambda)^*.$$

Proof Combining Theorem 7.5 and part 2 of Theorem 7.7 with Corollary 7.64 gives

$$F_{V^*}(\mu, \lambda)^* = (1 \otimes M(\mu)) F_V(\lambda, \mu)$$

for some $M(\mu) \in \mathbf{C}[[\mathsf{q}^{-\langle \alpha_1, \mu \rangle}, \ldots, \mathsf{q}^{-\langle \alpha_r, \mu \rangle}]] \otimes \mathsf{End}\, V[0]$, and

$$F_V(\lambda, \mu) = (\tilde{M}(\lambda) \otimes \mathsf{Id}) F_{V^*}(\mu, \lambda)$$

for some $\tilde{M}(\lambda) \in \mathbf{C}[[\mathsf{q}^{-\langle \alpha_1, \lambda \rangle}, \ldots, \mathsf{q}^{-\langle \alpha_r, \lambda \rangle}]] \otimes \mathsf{End}\, V[0]$. Then $\tilde{M}(\lambda) \otimes M(\mu) = \mathsf{Id}$, so $M(\mu)$ is independent of μ. Comparing the highest terms of $F_V(\lambda, \mu)$ and $F_{V^*}(\mu, \lambda)$ (they are both $\mathsf{q}^{-2\langle \lambda, \mu \rangle}$), we conclude that $M(\mu) = \mathsf{Id}$, and this proves the result. □

8

TRACES OF INTERTWINERS AND MACDONALD POLYNOMIALS

8.1 Macdonald polynomials

Consider $\mathfrak{g} = \mathfrak{sl}_n$ with weight lattice P. If $\lambda = (\lambda_1, \ldots, \lambda_n)$, then let $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. Then $\mathbf{C}[\mathsf{P}]$ is the ring generated by the monomials of the form

$$x^\lambda, \lambda_i - \lambda_j \in \mathbf{Z}, \sum_i^n \lambda_i = 0,$$

where $x_i = e^{\varepsilon_i}$, and $\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$ is the subring of $\mathfrak{S}_n$-invariant polynomials, where $\mathfrak{S}_n$ acts by permuting the x_i.

Definition 8.1 *The monomial symmetric function m_λ is given by*

$$m_\lambda = \sum_{\mu \in \mathfrak{S}_n \lambda} x^\mu.$$

Claim 8.2 $\{m_\lambda : \lambda \in \mathsf{P}_+\}$ *is a basis of* $\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$.

Definition 8.3 *The Macdonald operator $\mathcal{M}_1 : \mathbf{C}[\mathsf{P}] \to \mathbf{C}[\mathsf{P}]$ is defined as follows:*

$$\mathcal{M}_1 = \sum_{i=1}^n \prod_{j \neq i} \left(\frac{\mathsf{t} x_i - \mathsf{t}^{-1} x_j}{x_i - x_j} \right) T_i,$$

where $T_i f(x_1, \ldots, x_n) = f(x_1, \ldots, \mathsf{q}^2 x_i, \ldots, x_n), \mathsf{t}, \mathsf{q} \in \mathbf{C}, \mathsf{t} = \mathsf{q}^k$. (For simplicity we will restrict to the case $k \in \mathbb{Z}_+$.)

Proposition 8.4 $\mathcal{M}_1(\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}) \subset \mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$.

Proof It is checked directly that for every symmetric polynomial f, the residue of $\mathcal{M}_1 f$ at the hyperplane $x_i = x_j$ is zero. Hence the result. □

Proposition 8.5

$$\mathcal{M}_1 m_\lambda = c_{\lambda\lambda} m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu$$

for some $c_{\lambda\mu}$, with $c_{\lambda\lambda} = \sum_{i=1}^n \mathsf{q}^{2\lambda_i} \mathsf{t}^{2\rho_i}$.

Here $\mu < \lambda$ means $\lambda - \mu = \sum_{\alpha \in \mathsf{R}_+} k_\alpha \alpha, k_\alpha \geq 0$, and $\lambda \neq \mu$. Also $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right)$.

Proof

$$\begin{aligned}
&\mathcal{M}_1 m_\lambda \\
&= \sum_{i=1}^{n} \prod_{j=1}^{i-1} \left(\mathsf{t} - \mathsf{t}^{-1} \frac{x_j}{x_i}\right) \left(\sum_{k=0}^{\infty} \left(\frac{x_j}{x_i}\right)^k\right) \prod_{j=i+1}^{n} \left(-\mathsf{t}\frac{x_i}{x_j} + \mathsf{t}^{-1}\right) \sum_{k=0}^{\infty} \left(\frac{x_i}{x_j}\right)^k T_j m_\lambda \\
&= \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} \mathsf{t}\right) \left(\prod_{j=i+1}^{n} \mathsf{t}^{-1}\right) \mathsf{q}^{2\lambda_i} m_\lambda + \text{lower order terms} \\
&= c_{\lambda\lambda} m_\lambda + \text{lower order terms.}
\end{aligned}$$

□

Proposition 8.6 *There exists a unique basis* $\{P_{\lambda,k,\mathsf{q}} \mid \lambda \in \mathsf{P}_+\}$ *of* $\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$ *of the form*

$$P_{\lambda,k,\mathsf{q}} = m_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu}(\mathsf{q}, k) m_\mu$$

such that

$$\mathcal{M}_1 P_{\lambda,k,\mathsf{q}} = c_{\lambda\lambda}(\mathsf{q}, k) P_{\lambda,k,\mathsf{q}},$$

and $d_{\lambda,\mu}$ *is a rational function of* q.

Proof We note that for $\lambda \in \mathsf{P}_+$, the $\lambda_i + k\rho_i$ form a decreasing sequence; hence, given

$$c_{\lambda\lambda}(\mathsf{q}) = \sum_{i=1}^{n} \mathsf{q}^{2\lambda_i} \mathsf{t}^{2\rho_i} = \sum_{i=1}^{n} \mathsf{q}^{2(\lambda_i + k\rho_i)}$$

as a rational function in q, we can uniquely determine λ.

Thus, for generic q (or over $\mathbf{C}(\theta)$, where θ is a fixed nth root of q), all of the $c_{\lambda\lambda}$ are distinct; but the $c_{\lambda\lambda}$ are the eigenvalues of $\mathcal{M}_1$, so the result follows.

□

Definition 8.7 *The polynomials* $P_{\lambda,k,\mathsf{q}}$ *from Proposition 8.6 are known as the Macdonald polynomials.*

Definition 8.8 *For* $k \in \mathbf{Z}_+$, *the Macdonald denominator* $\Delta_{\mathsf{q},k}$ *is*

$$\Delta_{\mathsf{q},k} = \sum_{m=0}^{k-1} \prod_{\alpha \in \mathsf{R}_+} \left(1 - \mathsf{q}^{2m} e^{-\alpha}\right) = \sum_{m=0}^{k-1} \prod_{i<j} \left(1 - \mathsf{q}^{2m} \frac{x_i}{x_j}\right).$$

Definition 8.9 *We define a symmetric inner product on* $\mathbf{C}[\mathsf{P}]$ *as follows:*

$$\langle f, g \rangle_k = \text{constant term of } f\overline{g}\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}},$$

where the "constant term" of $h = \sum_\lambda c_\lambda m_\lambda$ *is* c_0*, and* $\overline{e^\lambda} = e^{-\lambda}, \overline{\mathsf{q}} = \mathsf{q}, \overline{g}(x) = g(x^{-1})$*. (Here* $(x_1, \ldots, x_n)^{-1} := (x_1^{-1}, \ldots, x_n^{-1})$*.) It can be shown that this form is nondegenerate if* q *is not a root of unity. This is clearly so for generic* q*.*

Proposition 8.10 $\mathcal{M}_1$ *is self-adjoint under* $\langle \cdot, \cdot \rangle_k$*.*

Proof Note that $\langle f, g \rangle_k = \left\langle f, g\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}} \right\rangle_0$. Now, the adjoints of $h(x)$ (multiplication by $h(x)$) and of T_i with respect to $\langle \cdot, \cdot \rangle_0$ are $h(x^{-1})$ and T_i, respectively.

Let us now compute adjoints with respect to $\langle \cdot, \cdot \rangle_k$. It is clear that $h(x)^* = h(x^{-1})$; furthermore,

$$\begin{aligned} \langle T_i f, g \rangle_k &= \left\langle T_i f, g\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}} \right\rangle_0 \\ &= \left\langle f, T_i g \cdot T_i \Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}} \right\rangle_0 \\ &= \left\langle f, \frac{T_i\left(\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}}\right)}{\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}}} \cdot T_i g \right\rangle_k, \end{aligned}$$

and this means that

$$T_i^* = \frac{T_i\left(\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}}\right)}{\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}}} \cdot T_i.$$

Now, it is easy to compute the following:

$$\frac{T_i\left(\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}}\right)}{\Delta_{\mathsf{q},k}\overline{\Delta_{\mathsf{q},k}}} = \prod_{j \neq l} \left(\frac{1 - \mathsf{q}^{2k} x_l / x_j}{1 - x_l / x_j} \right) \cdot \prod_{i \neq l} \left(\frac{1 - x_i / \mathsf{q}^2 x_l}{1 - \mathsf{q}^{2k-2} x_i / x_l} \right). \qquad (8.1)$$

We can now compute the adjoint of $\mathcal{M}_1$:

$$
\begin{aligned}
\mathcal{M}_1^* &= \sum_{l=1}^{n} \left(\prod_{j\neq l} \left(\frac{\mathsf{q}^k x_l - \mathsf{q}^{-k} x_j}{x_l - x_j} \right) T_l \right)^* \\
&= \sum_{l=1}^{n} \frac{T_l \Delta_{\mathsf{q},k} \overline{\Delta_{\mathsf{q},k}}}{\Delta_{\mathsf{q},k} \overline{\Delta_{\mathsf{q},k}}} \cdot T_l \prod_{j\neq l} \frac{\mathsf{q}^k x_l^{-1} - \mathsf{q}^{-k} x_j^{-1}}{x_l^{-1} - x_j^{-1}} \\
&= \sum_{l=1}^{n} \frac{T_l \Delta_{\mathsf{q},k} \overline{\Delta_{\mathsf{q},k}}}{\Delta_{\mathsf{q},k} \overline{\Delta_{\mathsf{q},k}}} \cdot T_l \prod_{j\neq l} \frac{\mathsf{q}^k x_j - \mathsf{q}^{-k} x_l}{x_j - x_l} \\
&= \sum_{l=1}^{n} \frac{T_l \Delta_{\mathsf{q},k} \overline{\Delta_{\mathsf{q},k}}}{\Delta_{\mathsf{q},k} \overline{\Delta_{\mathsf{q},k}}} \prod_{j\neq l} \frac{\mathsf{q}^k x_j - \mathsf{q}^{-k+2} x_l}{x_j - \mathsf{q}^2 x_l} \cdot T_l \\
&= \sum_{l=1}^{n} \prod_{j\neq l} \mathsf{q}^{-k} \frac{x_j - \mathsf{q}^{2k} x_l}{x_j - x_l} \cdot T_l \quad \text{(using (8.1))} \\
&= \sum_{l=1}^{n} \left(\prod_{j\neq l} \frac{\mathsf{q}^k x_l - \mathsf{q}^{-k} x_j}{x_l - x_j} \right) T_l \\
&= \mathcal{M}_1,
\end{aligned}
$$

so indeed $\mathcal{M}_1$ is self-adjoint. □

Corollary 8.11 $\langle P_{\lambda,k,\mathsf{q}}, P_{\mu,k,\mathsf{q}} \rangle_k = 0$ *whenever* $\lambda \neq \mu$.

Corollary 8.12 *The polynomials* $P_{\lambda,k,\mathsf{q}}$ *can be obtained by Gram–Schmidt orthogonalization of* m_λ *under* $\langle \cdot, \cdot \rangle_k$.

Remark 8.13 For this, we need to extend the partial ordering $\mu < \lambda$ to a total ordering $\mu \ll \lambda$. Since $\mathcal{M}_1$ is self-adjoint, from what we have shown it follows that the result of the Gram-Schmidt orthogonalization will not depend on the choice of $\ll$, and will always yield $P_{\lambda,k,\mathsf{q}}$.

Definition 8.14 *The higher Macdonald operators are defined as follows:*

$$
\mathcal{M}_r = \sum_{\substack{I\subset\{1,\ldots,n\} \\ |I|=r}} \prod_{i\in I, j\notin I} \frac{\mathsf{t} x_i - \mathsf{t}^{-1} x_j}{x_i - x_j} T_I,
$$

where $I = \{i_1, \ldots, i_r\} \Longrightarrow T_I = T_{i_1} \cdots T_{i_r}$.

We can prove the following results for $\mathcal{M}_r$; the proofs are similar to those of the corresponding results for $\mathcal{M}_1$.

Proposition 8.15 $\mathcal{M}_r(\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}) \subset \mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$.

Proposition 8.16

$$\mathcal{M}_r m_\lambda = c^{(r)}_{\lambda\lambda} m_\lambda + \sum_{\mu<\lambda} c^{(r)}_{\lambda\mu} m_\mu, \quad \textit{where } c^{(r)}_{\lambda\lambda} = \sum_{|I|=r} \prod_{i\in I} \mathsf{q}^{2\lambda_i} \mathsf{t}^{2\rho_i}.$$

Proposition 8.17 *$\mathcal{M}_r$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_k$.*

Corollary 8.18 *The same Macdonald polynomials $P_{\lambda,k,\mathsf{q}}$ are eigenvectors of $\mathcal{M}_r$:*

$$\mathcal{M}_r P_{\lambda,k,\mathsf{q}} = c^{(r)}_{\lambda\lambda} P_{\lambda,k,\mathsf{q}}.$$

Proof By Proposition 8.16, we see that $\mathcal{M}_r$ is triangular with respect to $\{P_{\lambda,k,\mathsf{q}}\}$; but by Corollary 8.11 and Proposition 8.17, $\mathcal{M}_r$ must by diagonal with respect to $\{P_{\lambda,k,\mathsf{q}}\}$. □

Corollary 8.19

$$[\mathcal{M}_r, \mathcal{M}_s] = 0 \quad \textit{for all } r, s.$$

Proof By Corollary 8.18, $\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$ has a basis consisting of common eigenvectors of $\mathcal{M}_r$ and $\mathcal{M}_s$. Thus $[\mathcal{M}_r, \mathcal{M}_s]$ is zero on $\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$. But it can be shown that a difference operator with rational coefficients that vanishes on $\mathbf{C}[\mathsf{P}]^{\mathfrak{S}_n}$ must vanish on $\mathbf{C}[\mathsf{P}]$, so the result follows. □

8.2 Vector-valued characters

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, and let $\lambda \in \mathsf{P}_+$. Let $L_\lambda = M_\lambda/J_\lambda$, an irreducible finite-dimensional representation. If V is a finite-dimensional representation, then we saw that for generic μ and fixed $\beta \in \mathsf{P}$,

$$\mathsf{Hom}_{\mathfrak{g}}(M_{\mu+\beta}, M_\mu \otimes V) \cong V[\beta],$$

with $\Phi \mapsto \langle\Phi\rangle$ under this isomorphism.

Lemma 8.20 *Let Y be a finite-dimensional representation of $\mathfrak{g}$. Then,*

$$\mathsf{Hom}_{\mathfrak{g}}(L_\nu, Y) = \mathsf{Hom}_{\mathfrak{g}}(M_\nu, Y).$$

Proof Since Y is finite-dimensional, any homomorphism $M_\nu \to Y$ must map $J_\nu \to 0$, and hence induces a homomorphism $L_\nu \to Y$. □

Lemma 8.21 *L^*_μ is generated by $\mathbf{v}^*_\mu$ over $\mathfrak{U}(\mathfrak{n}_+)$ with defining relations*

$$\mathsf{e}_i^{\langle\mu,\alpha_i^\vee\rangle+1}\mathbf{v}^*_\mu = 0.$$

Proof In M_μ, we have Verma submodules $M_{s_i(\mu+\rho)-\rho_i} = \left\langle \mathsf{f}_i^{\mu_i+1}\mathbf{v}_\mu\right\rangle$. We also have $J_\mu = \sum_{i=1}^{\mathsf{rank}\,\mathfrak{g}} M_{s_i(\mu+\rho)-\rho_i}$. This means that L_μ is generated over $\mathfrak{U}(\mathfrak{n}_-)$ by relations $\mathsf{f}_i^{\mu_i+1}\mathbf{v}_\mu = 0$, and the result follows. □

Theorem 8.22 *Suppose $\mu, \mu+\beta \in \mathsf{P}_+$. Then,*

$$\mathsf{Hom}_\mathfrak{g}(L_{\mu+\beta}, L_\mu \otimes V) \cong V[\beta]_\mu,$$

where $V[\beta]_\mu \stackrel{\text{def}}{=} \left\{ v \in V[\beta] \,|\, \mathsf{e}_i^{\langle\mu,\alpha_i^\vee\rangle+1} v = 0 \right\}$.

Proof We have

$$\begin{aligned} \mathsf{Hom}_\mathfrak{g}(L_{\mu+\beta}, L_\mu \otimes V) &= \mathsf{Hom}_\mathfrak{g}(M_{\mu+\beta}, L_\mu \otimes V) \quad \text{by Lemma 8.20} \\ &= \mathsf{Hom}_{\mathfrak{b}_+}((\mu+\beta), L_\mu \otimes V) \quad \text{by Frobenius reciprocity} \\ &= \mathsf{Hom}_{\mathfrak{b}_+}((\mu+\beta) \otimes L^*_\mu, V). \end{aligned} \tag{8.2}$$

Using Lemma 8.21, we see that $(\mu+\beta)\otimes L^*_\mu$ is generated by $\mathbf{v}^*_\mu$ with relations $\mathsf{e}_i^{\langle\mu,\alpha_i^\vee\rangle+1}\mathbf{v}^*_\mu = 0$ and $\mathsf{h}\mathbf{v}^*_\mu = \beta(\mathsf{h})\mathbf{v}^*_\mu$ for all $\mathsf{h} \in \mathfrak{h}$. Thus (8.2) becomes

$$\begin{aligned} \mathsf{Hom}_\mathfrak{g}(L_{\mu+\beta}, L_\mu \otimes V) &= \mathsf{Hom}_{\mathfrak{b}_+}((\mu+\beta) \otimes L^*_\mu, V) \\ &= \left\{ v \in V[\beta] \,|\, \mathsf{e}_i^{\langle\mu,\alpha_i^\vee\rangle+1} v = 0 \right\} \\ &= V[\beta]_\mu. \end{aligned}$$

□

Remark 8.23 This theorem is also true for $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$, where q is not a root of unity. The proof is the same.

For the remainder of this section, we will take $\mathfrak{g} = \mathfrak{sl}_n, V = S^{(k-1)n}\mathbf{C}^n$ (symmetric power), where k is a fixed integer. Then $V = L_{(k-1)n\omega_1}$, and a realization of V is

$$L_{(k-1)n\omega_1} = \{\text{homogeneous polynomials in } x_1 \cdots x_n, \text{ of degree } (k-1)n\},$$

with the following action of $\mathfrak{sl}_n$:

$$\mathsf{e}_i \mapsto x_i \frac{\partial}{\partial x_{i+1}}, \quad \mathsf{f}_i \mapsto x_{i+1}\frac{\partial}{\partial x_i}, \quad \mathsf{h}_i \mapsto x_i\frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}}.$$

A highest weight vector is then $x_1^{(k-1)n}$, and $V[0] = \mathbf{C}u$, where $u = (x_1\cdots x_n)^{k-1}$. Also, the Weyl group $\mathbf{W}$ of $\mathfrak{g}$ is then $\mathfrak{S}_n$.

Now

$$\mathsf{e}_i^p u = x_i^p \frac{\partial^p}{\partial x_{i+1}^p}(x_1 \cdots x_n)^{k-1} \begin{cases} \neq 0 & \text{if } p \leq k-1 \\ 0 & \text{if } p \geq k \end{cases}$$

Then

$$\begin{aligned} V[0]_\mu \neq 0 &\iff V[0]_\mu = V[0] \\ &\iff \mathsf{e}_i^{\mu_i+1} u = 0 \quad \text{for all } i \\ &\iff \mu_{i+1} \geq k \quad \text{for all } i \\ &\iff \mu - (k-1)\sum_i \omega_i = \mu - (k-1)\rho \in \mathsf{P}_+. \end{aligned}$$

Therefore,

$$\mathsf{Hom}_{\mathfrak{sl}_n}(L_{\mu+\beta}, L_\mu \otimes V) = \begin{cases} \mathbf{C} & \text{if } \mu - (k-1)\rho \in \mathsf{P}_+ \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8.24 *Let $\mu \in \mathsf{P}_+$. The vector-valued character of $L_{\mu+(k-1)\rho}$, $\varphi_\mu(\lambda) : V[0] \to V[0]$ is defined as follows:*

$$\begin{aligned} \varphi_\mu(\lambda)(u) &\overset{\text{def}}{=} \mathsf{tr}|_{L_{\mu+(k-1)\rho}} \left(\Phi_\mu^u e^{\bar{\lambda}}\right) \\ &= \sum_{\beta \in \mathsf{P}} \mathsf{tr}|_{L_{\mu+(k-1)\rho}[\beta]} \left(\Phi_\mu^u\right) e^{\langle \lambda, \beta \rangle}, \end{aligned}$$

where for $u \in V[0]$, $\Phi_\mu^u : L_{\mu+(k-1)\rho} \to L_{\mu+(k-1)\rho} \otimes V$ is an intertwiner.

Note that $V[0]$ is one-dimensional and thus can be identified with $\mathbf{C}$ by fixing a nonzero $u \in V[0]$; $\varphi_\mu(\lambda)$ then becomes a scalar-valued function.

Example 8.25 For $k = 1, V = \mathbf{C}$, we have

$$\varphi_\mu(\lambda) = \chi_{L_\mu}(\lambda) = \sum_{\beta \in \mathsf{P}} \dim L_\mu[\beta] e^{\langle \lambda, \beta \rangle}.$$

Lemma 8.26 *Let $\{v_\beta, \beta \in \mathsf{Q}\}$ be a basis of V. Let $X_i = \mathsf{F}_i \mathsf{q}^{\frac{d_i \mathsf{h}_i}{2}}$, and let $P \in \mathbf{C}\langle X_1, \cdots, X_{n-1}\rangle$ be a (noncommutative) polynomial of weight $-\beta$, where $\beta \in \mathsf{Q}_+$. Let $\varphi_\mu^P(\lambda) = \mathsf{tr}|_{L_{\mu+(k-1)\rho}} \left(\Phi_\mu^u P e^{\bar{\lambda}}\right)$. Then there exists a polynomial $Q_P \in \mathbf{C}(\mathsf{q})[\mathsf{P}]$ such that*

$$\varphi_\mu^P(\lambda) = \frac{Q_P \varphi_\mu(\lambda)}{\prod_{0<\gamma\leq\beta}(1 - \mathsf{q}^{-\langle \gamma,\gamma \rangle} e^{-\langle \lambda,\gamma \rangle})} v_{-\beta}. \tag{8.3}$$

Proof The proof will be by induction on β. For $\beta = 0$, the result is trivial. Suppose that the result is true whenever $\beta < \beta'$, and suppose P has weight $-\beta'$. Then

$$\mathbf{\Delta}(P) = P \otimes \mathsf{q}^{\mathsf{h}_{\beta'}} + \text{terms of higher first-component weight.}$$

Thus

$$\begin{aligned}\varphi_\mu^P(\lambda) &= \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\mathbf{\Delta}(P)\Phi_\mu^u e^{\bar{\lambda}}\right)\\ &= \mathsf{q}^{\mathsf{h}_{\beta'}}\ \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left((P\otimes \mathsf{Id})\Phi_\mu^u e^{\bar{\lambda}}\right) + Q'_P\\ &= \mathsf{q}^{-\left\langle \beta',\beta'\right\rangle}\ \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\Phi_\mu^u e^{\bar{\lambda}} P\right) + Q'_P\\ &= \mathsf{q}^{-\left\langle \beta',\beta'\right\rangle} e^{-\left\langle \lambda,\beta'\right\rangle}\varphi_\mu^P(\lambda) + Q'_P,\end{aligned}$$

where Q'_P is of the form required by equation (8.3), by the induction hypothesis. We then get

$$\varphi_\mu^P(\lambda) = \frac{1}{1-\mathsf{q}^{-\langle \beta',\beta'\rangle}e^{-\langle\lambda,\beta'\rangle}}Q'_P,$$

and the lemma follows. □

Lemma 8.27 *With the same notation as in Theorem 8.26, let us assume that* $\beta \in \mathsf{R}_+$ *and* $\lim_{\mathsf{q}\to 1} P = \mathsf{f}_\beta$*. Write* $P = P_\beta$*. Then* $Q_{P_\beta^{k-1}}$ *is a nonzero polynomial relatively prime to* $\prod_{j=1}^{k-1}(1-\mathsf{q}^{-2j}e^{-\bar{\beta}})$.

Proof It is enough to prove the lemma in the case $\mathsf{q} = 1$. In this case, we have $P_\beta = \mathsf{f}_\beta$, and then

$$\begin{aligned}\varphi_\mu^{P_\beta^{k-1}}(\lambda) &= \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\Phi_\mu^u \mathsf{f}_\beta^{k-1} e^{\bar{\lambda}}\right)\\ &= \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\mathbf{\Delta}(\mathsf{f}_\beta)\Phi_\mu^u \mathsf{f}_\beta^{k-2} e^{\bar{\lambda}}\right)\\ &= \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left((\mathsf{f}_\beta\otimes \mathsf{Id})\Phi_\mu^u \mathsf{f}_\beta^{k-2} e^{\bar{\lambda}}\right) + \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left((\mathsf{Id}\otimes \mathsf{f}_\beta)\Phi_\mu^u \mathsf{f}_\beta^{k-2} e^{\bar{\lambda}}\right)\\ &= \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\Phi_\mu^u \mathsf{f}_\beta^{k-2} e^{\bar{\lambda}}\mathsf{f}_\beta\right) + \mathsf{f}_\beta\ \mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\Phi_\mu^u \mathsf{f}_\beta^{k-2} e^{\bar{\lambda}}\right)\\ &= e^{-\langle\lambda,\beta\rangle}\varphi_\mu^{P^{k-1}}(\lambda) + \mathsf{f}_\beta\varphi_\mu^{P^{k-2}}(\lambda),\end{aligned}$$

and thus

$$\varphi_\mu^{P_\beta^{k-1}}(\lambda) = \left(1-e^{-\langle\lambda,\beta\rangle}\right)\mathsf{f}_\beta\varphi_\mu^{P_\beta^{k-2}}(\lambda).$$

Iterating, we obtain

$$\varphi_\mu^{P_\beta^{k-1}}(\lambda) = \left(1-e^{-\langle\lambda,\beta\rangle}\right)^{1-k}\mathsf{f}_\beta^{k-1}\varphi_\mu(\lambda).$$

Using Lemma 8.26, we see that for some constant $c \neq 0$,

$$
\begin{aligned}
Q_{P_\beta^{k-1}} &= c\frac{\prod_{0<\gamma\leq(k-1)\beta}(1-e^{-\langle\gamma,\lambda\rangle})}{(1-e^{-\langle\beta,\lambda\rangle})^{k-1}} \\
&= c\prod_{\substack{0<\gamma\leq(k-1)\beta\\ \gamma\neq s\beta}}(1-e^{-\langle\gamma,\lambda\rangle})\prod_{s=1}^{k-1}\left(1+e^{-\langle\beta,\lambda\rangle}+\cdots+e^{-(s-1)\langle\beta,\lambda\rangle}\right).
\end{aligned}
$$

The result follows. □

Lemma 8.28 *φ_μ is divisible by*

$$\prod_{\alpha\in\mathsf{R}_+}\prod_{m=1}^{k-1}\left(1-\mathsf{q}^{-2m}e^{-\bar{\alpha}}\right).$$

Proof For any $\alpha\in\mathsf{R}_+$, we have, by Lemma 8.26,

$$\varphi_\mu^{P_\alpha^{k-1}}(\lambda)=\frac{Q_{P_\alpha^{k-1}}\varphi_\mu(\lambda)}{\prod_{j=1}^{k-1}\left(1-\mathsf{q}^{-2j}e^{-\langle\alpha,\lambda\rangle}\right)G},$$

where G is a product of binomial factors. By Lemma 8.27, $Q_{P_\alpha^{k-1}}$ is relatively prime to $\prod_{j=1}^{k-1}\left(1-\mathsf{q}^{-2j}e^{-\langle\alpha,\lambda\rangle}\right)$, and the result follows. □

Theorem 8.29 (Etingof and Kirillov (1994))

$$\varphi_0(\lambda)=\prod_{\alpha\in\mathsf{R}_+}\prod_{m=1}^{k-1}\left(e^{\frac{\langle\alpha,\lambda\rangle}{2}}-\mathsf{q}^{-2m}e^{-\frac{\langle\alpha,\lambda\rangle}{2}}\right).$$

Proof The vector-valued character φ_0 of $L_{(k-1)\rho}$ satisfies

$$
\begin{aligned}
\varphi_0(\lambda) &= \mathsf{tr}|_{L_{(k-1)\rho}}\left(\Phi_0e^{\bar{\lambda}}\right) \\
&= \prod_{\alpha\in\mathsf{R}_+}\prod_{m=1}^{k-1}\left(e^{\frac{\langle\alpha,\lambda\rangle}{2}}-\mathsf{q}^{-2m}e^{-\frac{\langle\alpha,\lambda\rangle}{2}}\right)\cdot P(\lambda) \qquad (8.4)
\end{aligned}
$$

for some Laurent polynomial $P(\lambda)$, by Lemma 8.28. Now, the term of highest weight in (8.4) has weight $(k-1)\rho$, while the highest weight in the product that precedes $P(\lambda)$ in (8.4) is

$$(k-1)\sum_{\alpha\in\mathsf{R}_+}\frac{\alpha}{2}=(k-1)\rho.$$

Thus, the only dominant weight γ for which $P(\lambda)$ can contain a nonzero term $b_\gamma e^{\langle\gamma,\lambda\rangle}$ is $\gamma=0$. In a similar way, we see that the same statement is true of

$P(w\lambda)$, for $w \in \mathbf{W} = \mathfrak{S}_n$, which means that $P(\lambda)$ must be a constant. But the highest term in $\varphi_0(\lambda)u$ is $1 \cdot e^{(k-1)\langle\rho,\lambda\rangle}$; hence, $P(\lambda) = 1$, and the result follows. □

Theorem 8.30 (Etingof and Kirillov (1994)) *$\varphi_\mu(\lambda)$ is divisible by $\varphi_0(\lambda)$, for all $\mu \in \mathsf{P}_+$.*

Proof This follows immediately from Lemma 8.28 and Theorem 8.29. □

Theorem 8.31 (Etingof and Kirillov (1994)) *The polynomial $P'_\mu \overset{\text{def}}{=} \frac{\varphi_\mu}{\varphi_0}$ is symmetric, for all $\mu \in \mathsf{P}_+$.*

Proof Consider

$$\mathsf{Id} \otimes \Phi_0 : L_\mu \otimes L_{(k-1)\rho} \to L_\mu \otimes L_{(k-1)\rho} \otimes V.$$

We have

$$\begin{aligned} \mathsf{tr}\left((\mathsf{Id} \otimes \Phi_0)\, e^\lambda\big|_{L_\mu \otimes L_{(k-1)\rho}} \right) &= \mathsf{tr}|_{L_\mu}\left(e^\lambda\right) \cdot \mathsf{tr}|_{L_{(k-1)\rho}}\left(\Phi_0 e^\lambda\right) \\ &= \chi_{L_\mu}(\lambda)\varphi_0(\lambda)u. \end{aligned} \tag{8.5}$$

On the other hand,

$$L_\mu \otimes L_{(k-1)\rho} = L_{\mu+(k-1)\rho} \oplus \left(\oplus_{\nu<\mu} L_{\nu+(k-1)\rho} \otimes N_\nu\right),$$

where

$$N_\nu = \mathsf{Hom}_{\mathfrak{U}_\mathsf{q}(\mathfrak{g})}\left(L_{\nu+(k-1)\rho}, L_\mu \otimes L_{(k-1)\rho}\right).$$

Now, writing

$$(\mathsf{Id} \otimes \Phi_0)|_{L_{\nu+(k-1)\rho} \otimes N_\nu} = \Phi_\nu \otimes X_\nu,$$

for some $X_\nu : N_\nu \to N_\nu$, we see that

$$\mathsf{tr}\left((\mathsf{Id} \otimes \Phi_0)\, e^\lambda\right) = \left(\varphi_\mu(\lambda) + \sum_{\nu<\mu} (\mathsf{tr}\, X_\nu)\varphi_\nu(\lambda)\right) u. \tag{8.6}$$

Combining (8.5) and (8.6), we get

$$\begin{aligned} \chi_\mu(\lambda)\varphi_0(\lambda) &= \varphi_\mu(\lambda) + \sum_{\nu<\mu} (\mathsf{tr}\, X_\nu)\varphi_\nu(\lambda) \\ \Longrightarrow \chi_\mu(\lambda) &= P'_\mu(\lambda) + \sum_{\nu<\mu} (\mathsf{tr}\, X_\nu) P'_\nu(\lambda). \end{aligned} \tag{8.7}$$

We can now prove by induction that P'_μ is symmetric for all $\mu \in \mathsf{P}_+$. For $\mu = 0$, it is obvious, since $P'_0 = 1$; suppose $P'_\nu(\lambda)$ is symmetric for all $\nu < \mu$; then by (8.7) we have

$$P'_\mu(\lambda) = \chi_\mu(\lambda) - \sum_{\nu<\mu} (\mathsf{tr}\, X_\nu) P'_\nu(\lambda),$$

which is symmetric by induction. This completes the proof. □

Theorem 8.32 (Etingof and Kirillov (1994)) *The polynomial φ_μ/φ_0 is the Macdonald polynomial:*

$$\frac{\varphi_\mu}{\varphi_0}(\lambda) = P_{\mu,\mathsf{q}^{-1},k}.$$

Proof Let us write

$$\varphi_\mu(\lambda) = e^{\langle \mu+(k-1)\rho,\lambda\rangle}\left(1+\sum_{\beta<0} e^{\langle\beta,\lambda\rangle} d_{\beta,\mu}\right),$$

for all μ. Then,

$$\frac{\varphi_\mu}{\varphi_0}(\lambda) = e^{\langle \mu,\lambda\rangle}\left(1+\sum_{\beta<0} e^{\langle\beta,\lambda\rangle} d'_{\beta,\mu}\right). \tag{8.8}$$

Now remember the Cartan involution ω, defined in Section 2.6. It is an anti-automorphism of coalgebras, and we have a natural isomorphism $Y^\omega \cong Y^*$.

We then have

$$\Phi_\mu^\omega : L^\omega_{\mu+(k-1)\rho} \to \left(L_{\mu+(k-1)\rho}\otimes V\right)^\omega;$$

or equivalently,

$$\Phi_\mu^\omega : L^*_{\mu+(k-1)\rho} \to V^*\otimes L_{\mu+(k-1)\rho}.$$

We then consider the tensor product map

$$\Phi_\mu\otimes\Phi_\nu^\omega : L_{\mu+(k-1)\rho}\otimes L^*_{\nu+(k-1)\rho} \to L_{\mu+(k-1)\rho}\otimes V\otimes V^*\otimes L^*_{\nu+(k-1)\rho},$$

and we define

$$\Psi : L_{\mu+(k-1)\rho}\otimes L^*_{\nu+(k-1)\rho} \to L_{\mu+(k-1)\rho}\otimes L^*_{\nu+(k-1)\rho}$$

to be the composition

$$(\mathsf{Id}\otimes K_{V,V^*}\otimes\mathsf{Id})\circ(\Phi_\mu\otimes\Phi_\nu^\omega),$$

where $K_{V,V^*}(v\otimes v^*) = v^*(v)$.

Note that

$$\begin{aligned}
&\mathsf{tr}|_{L_{\mu+(k-1)\rho}\otimes L^*_{\nu+(k-1)\rho}}\,\Psi\boldsymbol{\Delta}(e^\lambda)\\
&= K_{V,V^*}\left(\mathsf{tr}|_{L_{\mu+(k-1)\rho}}\left(\Phi_\mu e^\lambda\right)\otimes\mathsf{tr}|_{L^*_{\nu+(k-1)\rho}}\left(\Phi_\nu^\omega e^{-\lambda}\right)\right)\\
&= \varphi_\mu(\lambda)\varphi_\nu(-\lambda).
\end{aligned} \tag{8.9}$$

We can now decompose

$$L_{\mu+(k-1)\rho}\otimes L^*_{\nu+(k-1)\rho} = \oplus_{\beta\in\mathsf{P}_+} N_{\mu\nu\beta}\otimes L_\beta,$$

and $\Psi = \sum_{\beta\in\mathsf{P}_+}\Psi_\beta\otimes\mathsf{Id}_{L_\beta}$ (since it is an intertwiner).

Therefore,

$$\mathsf{tr}\left(\Psi \mathbf{\Delta}(e^{\lambda})\right) = \sum_{\beta \in \mathsf{P}_+} \mathsf{tr}(\Psi_\beta)\, \mathsf{tr}_{L_\beta}(e^{\lambda}) = \sum_{\beta \in \mathsf{P}_+} \mathsf{tr}(\Psi_\beta) \chi_{L_\beta}(\lambda).$$

Then we have

$$\begin{aligned} \langle \varphi_\mu, \varphi_\nu \rangle &:= \frac{1}{n!} \,\mathsf{C.T.}\left(\varphi_\mu(\lambda)\varphi_\nu(-\lambda)\delta(\lambda)\delta(-\lambda)\right) \\ &= \frac{1}{n!}\,\mathsf{C.T.}\left(\sum_{\beta \in \mathsf{P}_+} \mathsf{tr}(\Psi_\beta)\chi_{L_\beta}(\lambda)\delta(\lambda)\delta(-\lambda) \right) \\ &= \frac{1}{n!} \sum_{\beta \in \mathsf{P}_+} \mathsf{tr}(\Psi_\beta)\,\mathsf{C.T.}\left(\chi_{L_\beta}(\lambda)\delta(\lambda)\delta(-\lambda)\right) \\ &= \frac{1}{n!} \sum_{\beta \in \mathsf{P}_+} \mathsf{tr}(\Psi_\beta)\,\langle 1, \chi_{L_\beta} \rangle_{\text{Weyl}} \\ &= \begin{cases} \mathsf{tr}(\Psi_0) & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu. \end{cases} \end{aligned}$$

(Here $\mathsf{C.T.}()$ denotes the operation of taking the constant term of a Laurent polynomial or series, and $\langle A, B \rangle_{\text{Weyl}} = \mathsf{C.T.}\left((A(\lambda)B(-\lambda)\delta(\lambda)\delta(-\lambda))\right)$ is the Weyl inner product.)

The last equality holds because if $\mu \neq \nu$, then $\mathbf{C} \not\subset L_{\mu+(k-1)\rho} \otimes L^*_{\nu+(k-1)\rho}$, hence $N_{\mu\nu 0} = 0$.

This proves that $\frac{\varphi_\mu}{\varphi_0}(\lambda)$ is the Macdonald polynomial. □

Let $x_i = \mathsf{q}^{2\lambda_i}$. Then the Macdonald operator $\mathcal{M}_r$ becomes

$$\mathcal{M}_r = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = r}} \prod_{i \in I, j \notin I} \frac{\mathsf{q}^k \mathsf{q}^{2\lambda_i} - \mathsf{q}^{-k}\mathsf{q}^{2\lambda_j}}{\mathsf{q}^{2\lambda_i} - \mathsf{q}^{2\lambda_j}} T_I,$$

where $I = \{i_1, \dots, i_r\} \Longrightarrow T_I = T_{i_1} \cdots T_{i_r}$ and $T_i \lambda_j = \lambda_j + \delta_{ij}$. We know that for generic $\mu \in \mathfrak{h}^*$, the spectrum of $\mathcal{M}_r$ on Y_μ is simple (note that this is *not* true for $\mu \in \mathsf{P}_+$), and there is a unique eigenfunction of the form

$$f_k(\mathsf{q}, \lambda, \mu) = \mathsf{q}^{2\langle \lambda, \mu - (k-1)\rho \rangle} \left(1 + \sum_{\beta < 0} c_\beta \mathsf{q}^{2\langle \beta, \lambda \rangle} \right)$$

such that

$$\mathcal{M}_r f_k = \left(\sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = r}} \prod_{i \in I} \mathsf{q}^{2(\mu+\rho)_i} \right) f_k.$$

Theorem 8.33

$$f_k(\mathsf{q}, \lambda, \mu) = \gamma_k(\mathsf{q}, \lambda)^{-1} \Psi_V(\mathsf{q}^{-1}, -\lambda, \mu),$$

where

$$\gamma_k(\mathsf{q}, \lambda) = \prod_{m=1}^{k-1} \prod_{l<j} \left(\mathsf{q}^{\lambda_l - \lambda_j} - \mathsf{q}^{2m} \mathsf{q}^{\lambda_j - \lambda_l} \right).$$

Proof This follows from the definition of Ψ_V and from theorems 8.29 and 8.32 (see Etingof and Kirillov (1994), Etingof and Varchenko (2000)). □

Corollary 8.34 (Felder and Varchenko (1997)) *Let* $\Lambda^r \mathbf{C}_{\mathsf{q}}^n$ *denote the* q*-analog of the* r*th fundamental representation of* $\mathfrak{sl}_n$*. Then,*

$$\mathbf{D}_{\Lambda^r \mathbf{C}_{\mathsf{q}}^n}(\mathsf{q}^{-1}, -\lambda) = \delta_{\gamma_k} \circ \mathcal{M}_r \circ (\delta_{\gamma_k})^{-1}.$$

Proof This follows from Theorem 8.33 and part 1 of Theorem 7.7 (see Etingof and Varchenko (2000)). □

9

DYNAMICAL WEYL GROUP

In this chapter we will review the theory of the dynamical Weyl group. This theory has many applications, in particular to study of traces of intertwining operators. The theory of the dynamical Weyl group was developed in Tarasov and Varchenko (2000), Etingof and Varchenko (2002).

9.1 Dynamical Weyl group (for $\mathfrak{g} = \mathfrak{sl}_2$)

Let $\mathfrak{g}$ be a simple Lie algebra, and let $\mathbf{W}$ denote its Weyl group. Given $\lambda \in \mathsf{P}_+$, we consider the Verma module M_λ. What are the singular vectors of M_λ (namely, the $v \in M_\lambda$ such that $\mathsf{E}_i v = 0$ for $i = 1, \ldots, r$)?

Let $w \cdot \lambda$ denote the shifted action of the Weyl group on $\mathfrak{h}^*$: $w \cdot \lambda = w(\lambda+\rho)-\rho$, for all $w \in \mathbf{W}$.

Theorem 9.1 (BGG theorem, Bernstein *et al.* (1971)) *Singular vectors exist only in weights $w \cdot \lambda, w \in \mathbf{W}$. In each of these weights, there exists one singular vector, up to scaling.*

Example 9.2 For $\mathfrak{sl}_2$, the singular vector $\frac{\mathsf{f}^{\lambda+1}}{(\lambda+1)!}\mathbf{v}_\lambda$ has weight $-\lambda-2$; indeed, we can take g to be the nontrivial element of $\mathbf{W}$, and then $g \cdot \lambda = g(\lambda+\rho) - \rho = g(\lambda+1) - 1 = -\lambda - 2$.

In this section, we will consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Let V be a representation of $\mathfrak{U}_\mathsf{q}(\mathfrak{sl}_2)$ and let $v \in V$, with $\nu = \mathsf{wt}\, v$. We will consider the restriction of $\Phi_\lambda^v : M_\lambda \to M_\lambda \otimes V$ to $M_{w'(\lambda+\rho)-\rho}$.

Claim 9.3 *For $\lambda \gg 0$, we have $\Phi_\lambda^v|_{M_{-\lambda-2}} \subset M_{-\lambda+\nu-2} \otimes V$.*

Proof We write

$$\Phi_\lambda^v(\mathsf{F}^{\lambda+1}\mathbf{v}_\lambda) = \sum_{k=0}^{\infty} c_k \mathsf{F}^k \mathbf{v}_{\lambda-\nu} \otimes w_k. \tag{9.1}$$

Consider the term with lowest power k_0 on the right-hand side of (9.1). For such k_0, $\mathsf{F}^{k_0}\mathbf{v}_{\lambda-\nu}$ is killed by E (to see this, just multiply both sides of (9.1) on the left by $\mathsf{E} \otimes \mathsf{q}^\mathsf{h} + \mathsf{Id} \otimes \mathsf{E}$). This means that k_0 is either 0 or $-\lambda+\nu-2$. But $k_0 = 0$ implies $\mathsf{wt}\, w_0 = -2\lambda - 2 + \nu \ll 0$ (since $\lambda \gg 0$), and this is impossible since V is finite-dimensional. Thus $k_0 = -\lambda + \nu - 2$, and the claim is proved. □

Knowing that

$$\Phi_\lambda^v : M_\lambda \to M_{\lambda-\nu} \otimes V$$

restricts to a map

$$\Phi_\lambda^v : M_{-\lambda-2} \to M_{-\lambda+\nu-2} \otimes V$$

for $\lambda \gg 0$, we can now define an operator

$$A_V(\lambda) : V \to V$$

such that

$$A_V(\lambda)v = \left\langle \Phi_\lambda^v|_{M_{-\lambda-2}} \right\rangle,$$

with $A_V(V[\nu]) \subset V[-\nu]$.

Proposition 9.4 *A_V is invertible for $\lambda \gg 0$.*

Proof Suppose $A_V(\lambda)|_{V[\nu]} = 0$. Then $\Phi_\lambda^v|_{M_{-\lambda-2}} = 0$. So Φ_λ^v induces a map

$$\Phi_\lambda^v : M_\lambda / M_{-\lambda-2} \to M_{\lambda-\nu} \otimes V.$$

But $M_\lambda/M_{-\lambda-2} = L_\lambda$, and thus we get a nonzero homomorphism

$$L_\lambda \otimes V^* \to M_{\lambda-\nu},$$

which is impossible since the Verma module $M_{\lambda-\nu}$ does not have a nonzero finite-dimensional submodule. □

Lemma 9.5 *Let U, V be representations of $\mathfrak{U}_\mathsf{q}(\mathfrak{sl}_2)$; then*

$$A_{U\otimes V}(\lambda) J_{UV}(\lambda) = J_{UV}(-\lambda-2) A_V^{(2)}(\lambda) A_U^{(1)}(\lambda - h^2). \tag{9.2}$$

Proof Let $u \in U[\mu], v \in V[\nu]$. We have

$$A_V^{(2)}(\lambda) A_U^{(1)}(\lambda - h^2)(u \otimes v) = \left\langle \Phi_{\lambda-\nu}^u|_{M_{-\lambda+\nu-2}} \right\rangle \otimes \left\langle \Phi_\lambda^v|_{M_{-\lambda-2}} \right\rangle,$$

and thus

$$\begin{aligned} J_{UV}(-\lambda-2) A_V^{(2)}(\lambda) A_U^{(1)}(\lambda-h^2)(u\otimes v) &= \left\langle (\Phi_\lambda^u \otimes \mathsf{Id}) \Phi_\lambda^v|_{M_{-\lambda-2}} \right\rangle \\ &= A_{U\otimes V}(\lambda) J_{UV}(\lambda)(u \otimes v). \end{aligned}$$

□

Proposition 9.6 *Let $V = \mathbf{C}^2 = \mathbf{C}v_+ \oplus \mathbf{C}v_-$. Then,*

$$A_V(\lambda) = \begin{pmatrix} 0 & -\mathsf{q}^{-1}\frac{[\lambda+2]_\mathsf{q}}{[\lambda+1]_\mathsf{q}} \\ \mathsf{q} & 0 \end{pmatrix}.$$

Proof We have

$$\Phi_\lambda^{v_+} : M_\lambda \to M_{\lambda-1} \otimes V,$$

with

$$\Phi_\lambda^{v_+}\mathbf{v}_\lambda = \mathbf{v}_{\lambda-1} \otimes v_+,$$

and hence

$$\begin{aligned}
&\Phi_\lambda^{v_+}\frac{\mathsf{F}^{\lambda+1}}{[\lambda+1]_\mathsf{q}!}\mathbf{v}_\lambda \\
&\quad= \frac{1}{[\lambda+1]_\mathsf{q}!}(\mathsf{F}\otimes 1 + \mathsf{q}^{-\mathsf{h}} \otimes \mathsf{F})^{\lambda+1}(\mathbf{v}_{\lambda-1} \otimes v_+) \\
&\quad= \frac{1}{[\lambda+1]_\mathsf{q}!}(\mathsf{F}^{\lambda+1}\mathbf{v}_{\lambda-1} \otimes v_+) \\
&\qquad+ \frac{1}{[\lambda+1]_\mathsf{q}!}\left(\mathsf{q}^{-\lambda+1} + \mathsf{q}^{-\lambda+3} + \cdots + \mathsf{q}^{\lambda+1}\right)(\mathsf{F}^\lambda \mathbf{v}_{\lambda-1} \otimes v_-) \\
&\quad= \frac{1}{[\lambda+1]_\mathsf{q}!}(\mathsf{F}^{\lambda+1}\mathbf{v}_{\lambda-1} \otimes v_+) + \frac{1}{[\lambda+1]_\mathsf{q}!}\frac{\mathsf{q}^{\lambda+1} - \mathsf{q}^{-\lambda-1}}{1-\mathsf{q}^{-2}}(\mathsf{F}^\lambda \mathbf{v}_{\lambda-1} \otimes v_-) \\
&\quad= \frac{1}{[\lambda+1]_\mathsf{q}!}(\mathsf{F}^{\lambda+1}\mathbf{v}_{\lambda-1} \otimes v_+) + \mathsf{q}\frac{1}{[\lambda]_\mathsf{q}!}(\mathsf{F}^\lambda \mathbf{v}_{\lambda-1} \otimes v_-).
\end{aligned}$$

Thus $A_V(\lambda)v_+ = \mathsf{q}v_-$. We also have

$$\Phi_\lambda^{v_-} : M_\lambda \to M_{\lambda+1} \otimes V,$$

with

$$\Phi_\lambda^{v_-}\mathbf{v}_\lambda = \mathbf{v}_{\lambda+1} \otimes v_- - \frac{\mathsf{q}^{-1}}{[\lambda+1]_\mathsf{q}}\mathsf{F}\mathbf{v}_{\lambda+1} \otimes v_+,$$

and hence

$$\begin{aligned}
&\Phi_\lambda^{v_-}\frac{\mathsf{F}^{\lambda+1}}{[\lambda+1]_\mathsf{q}!}\mathbf{v}_\lambda \\
&\quad= \frac{1}{[\lambda+1]_\mathsf{q}!}(\mathsf{F}\otimes 1 + \mathsf{q}^{-\mathsf{h}} \otimes \mathsf{F})^{\lambda+1}\left(\mathbf{v}_{\lambda+1} \otimes v_- - \frac{\mathsf{q}^{-1}}{[\lambda+1]_\mathsf{q}}\mathsf{F}\mathbf{v}_{\lambda+1} \otimes v_+\right) \\
&\quad= -\frac{1}{[\lambda+1]_\mathsf{q}!}\frac{\mathsf{q}^{-1}}{[\lambda+1]_\mathsf{q}}(\mathsf{F}^{\lambda+2}\mathbf{v}_{\lambda+1} \otimes v_+) + B \otimes v_- \quad \text{for some } B \\
&\quad= -\frac{[\lambda+2]_\mathsf{q}}{[\lambda+1]_\mathsf{q}}\frac{\mathsf{F}^{\lambda+2}\mathsf{q}^{-1}}{[\lambda+2]_\mathsf{q}!}\mathbf{v}_{\lambda+1} \otimes v_+ + B \otimes v_-.
\end{aligned}$$

Thus $A_V(\lambda)v_- = -\mathsf{q}^{-1}\frac{[\lambda+2]_\mathsf{q}}{[\lambda+1]_\mathsf{q}}v_+$. This completes the proof. □

In general, we let $V = V_m, m \in \mathbf{Z}_+$, and we write $A_m \stackrel{\text{def}}{=} A_{V_m}$. Consider

$$A_m^k(\lambda) : V_m[m-2k] \to V_m[2k-m].$$

Proposition 9.7 *We have*

$$A_m^k(\lambda) = c_m^k \prod_{j=1}^{k} \frac{[\lambda+1-j]_{\mathsf{q}}}{[\lambda-m+k+j]_{\mathsf{q}}},$$

where c_m^k is a nonzero constant (independent of λ).

Proof Let $U = V_1, V = V_m$ in (9.2); we get

$$A_{V_1 \otimes V_m}(\lambda) J_{V_1 V_m}(\lambda) = J_{V_1 V_m}(-\lambda-2) A_{V_m}^{(2)}(\lambda) A_{V_1}^{(1)}(\lambda - h^2). \tag{9.3}$$

We then take the determinant of both sides of (9.3), using $\mathsf{det} J = 1$ and $V_1 \otimes V_m = V_{m-1} \oplus V_{m+1}$. We get

$$A_{m+1}^0(\lambda) = A_m^0(\lambda) A_1^0(\lambda - m),$$

and, for $k > 0$,

$$A_{m+1}^k(\lambda) A_{m-1}^{k-1}(\lambda) = A_m^k(\lambda) A_m^{k-1}(\lambda) A_1^0(\lambda - m + 2k) A_1^1(\lambda - m + 2k - 2)$$

We then get

$$A_{m+1}^0(\lambda) = c_{m+1}^0 \quad \text{for some constant } c_{m+1}^0,$$

and, for $k > 0$,

$$A_{m+1}^k(\lambda) = \frac{[\lambda - m + 2k]_{\mathsf{q}}}{[\lambda - m + 2k - 1]_{\mathsf{q}}} A_m^k(\lambda) A_m^{k-1}(\lambda) A_{m-1}^{k-1}(\lambda)^{-1},$$

using Proposition 9.6. The result follows by induction on m. □

Corollary 9.8

1. *$A_V(\lambda)$ extends uniquely to a rational function of q^λ (of λ if $\mathsf{q} = 1$);*
2. *The following limits exist:*
   - $A_V^+ \stackrel{\text{def}}{=} \lim_{q^\lambda \to \infty} A_V(\lambda)$,
   - $A_V^- \stackrel{\text{def}}{=} \lim_{q^\lambda \to 0} A_V(\lambda)$;
3. $A_V^+\big|_{V_m[m-2k]} = \mathsf{q}^{-2k(m-k+1)} \, A_V^-\big|_{V_m[m-2k]}$;

4. If

$$A_V^\infty|_{V_m[m-2k]} \stackrel{\text{def}}{=} \mathsf{q}^{-k(m-k+1)} A_V^- = \mathsf{q}^{k(m-k+1)}\ A_V^+|_{V_m[m-2k]}\,,$$

then $A_V(\lambda) = A_V^\infty B_V(\lambda)$, *where*

$$B_V(\lambda)|_{V_m[m-2k]} : V_m[m-2k] \to V_m[m-2k]$$

is defined by

$$B_V(\lambda)|_{V_m[m-2k]} = \prod_{j=1}^{k} \frac{[\lambda+1+j]_{\mathsf{q}}}{[\lambda-m+k+j]_{\mathsf{q}}} \,\mathsf{Id}\,.$$

Proposition 9.9 *Let v_m be the highest weight vector of V_m, and let $v_{m-2j} = (\mathsf{F}^j/[j]_{\mathsf{q}})v_m$. Then $A_V^\infty v_{m-2j} = (-1)^j v_{2j-m}$.*

Proof See Etingof and Varchenko (2002). □

9.2 Dynamical Weyl group (for any finite-dimensional simple $\mathfrak{g}$)

Now, we want to generalize this to an arbitrary finite-dimensional simple Lie algebra $\mathfrak{g}$. Let $\mathbf{W}$ be the Weyl group of $\mathfrak{g}$, and let V be a finite-dimensional representation of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$. Let $\mathsf{q}_i = \mathsf{q}^{d_i}$, and for $\lambda \in \mathfrak{h}^*$, we will define

$$A_{s_i,V}(\lambda) \stackrel{\text{def}}{=} A_{V|_{\mathfrak{U}_{\mathsf{q}_i}(\mathfrak{sl}_{2_i})}}(\lambda(\mathsf{h}_i)).$$

Let $w \in W$, and let δ denote the reduced decomposition $s_{i_1}\cdots s_{i_l}$ of w.

Definition 9.10

$$A_{w,V,\delta}(\lambda) \stackrel{\text{def}}{=} A_{s_{i_1}}(s_{i_2}\cdots s_{i_l}\cdot\lambda)\cdots A_{s_{i_{l-1}}}(s_{i_l}\cdot\lambda)A_{s_{i_l}}(\lambda).$$

Denote $\alpha^j = s_{i_l}\cdots s_{i_{j-1}}(\alpha_{i_j})$, and $n_j = 2\frac{\langle\lambda+\rho,\alpha^j\rangle}{\langle\alpha^j,\alpha^j\rangle} > 0$.

Lemma 9.11

$$\mathbf{v}^\lambda_{w\cdot\lambda,\delta} \stackrel{\text{def}}{=} \frac{\mathsf{F}^{n_1}_{\alpha_{i_1}}\cdots\mathsf{F}^{n_l}_{\alpha_{i_l}}}{[n_1]!_{\mathsf{q}^{d_{i_1}}}\cdots[n_l]!_{\mathsf{q}^{d_{i_l}}}}\mathbf{v}_\lambda$$

is independent of the choice of reduced decomposition δ and is singular of weight $w\cdot\lambda$.

Proof The fact that this vector is singular is proved in a straightforward manner by induction in l (this is a rank 1 calculation). It is enough to prove that if δ,δ' are two reduced decompositions of w that differ by only one braid relation, then $\mathbf{v}^\lambda_{w\cdot\lambda,\delta} = \mathbf{v}^\lambda_{w\cdot\lambda,\delta'}$. In particular, we can focus on the rank two Lie algebras

A_2, B_2 and G_2. In this case, the maximal element $w_0 \in \mathbf{W}$ is the only one that does not have a unique reduced decomposition, so we can assume that $w = w_0$. Then in each reduced decomposition, each of the positive roots appears exactly once, which implies that

$$\{(n_1, d_1), \dots, (n_l, d_l)\} = \{(n'_1, d'_1), \dots, (n'_l, d'_l)\}.$$

Now let $u = \mathsf{F}_{\alpha_{i_1}}^{n_1} \cdots \mathsf{F}_{\alpha_{i_l}}^{n_l} \mathbf{v}_\lambda$ and $u' = \mathsf{F}_{\alpha_{i'_1}}^{n'_1} \cdots \mathsf{F}_{\alpha_{i'_l}}^{n'_l} \mathbf{v}_\lambda$. Then $u, u' \in M_\lambda$ are singular vectors of weight $w_0 \cdot \lambda$. Now, by the BGG theorem above, for A_2, B_2 and G_2, the space of singular vectors of weight $w_0 \cdot \lambda$ in M_λ is one-dimensional; hence, $u' = cu$ for some $c \in \mathbf{C} \setminus \{0\}$.

Let

$$F = \mathbf{C}[\mathsf{F}_1, \dots, \mathsf{F}_r] / \left\langle \mathsf{F}_i \mathsf{F}_j - \mathsf{q}_i^{a_{ij}} \mathsf{F}_j \mathsf{F}_i, i < j \right\rangle$$

and

$$\hat{F} = \mathbf{C}[\mathsf{F}_1, \dots, \mathsf{F}_r] / \left\langle \mathsf{F}_i \mathsf{F}_j - \mathsf{q}_i^{-a_{ij}} \mathsf{F}_j \mathsf{F}_i, i < j \right\rangle.$$

Let $\Psi : \mathfrak{U}_\mathsf{q}(\mathfrak{n}_-) \to F$ and $\hat{\Psi} : \mathfrak{U}_\mathsf{q}(\mathfrak{n}_-) \to \hat{F}$ be the natural homomorphisms. (It is easy to check that they are well-defined since the relations $\mathsf{F}_i \mathsf{F}_j = \mathsf{q}_i^{\pm a_{ij}} \mathsf{F}_j \mathsf{F}_i$ imply the q-Serre relations.) Then, for some m, we have $\Psi(u') = \mathsf{q}^m \Psi(u)$ and $\hat{\Psi}(u') = \mathsf{q}^{-m} \hat{\Psi}(u)$; thus $c = \mathsf{q}^m = \mathsf{q}^{-m}$. Therefore $c = 1$, and the result follows. □

Lemma 9.12 *Consider* $\Phi : M_\lambda \to M_\mu \otimes V, \lambda \in \mathsf{P}_+, \langle \lambda, \alpha_i \rangle \gg 0$. *Then*

$$\Phi \mathbf{v}^\lambda_{w \cdot \lambda, \delta} = \mathbf{v}^\mu_{w \cdot \mu, \delta} \otimes A_{w,V,\delta}(\lambda) \langle \Phi \rangle + \text{ lower order terms.}$$

Proof This follows easily from Definition 9.10, by induction on the length of w. □

Theorem 9.13 $A_{w,V,\delta}(\lambda)$ *is independent of the choice of* δ.

Proof For λ large dominant, this follows immediately from Lemmas 9.11 and 9.12. For arbitrary λ, this follows from the fact that $A_{w,V,\delta}(\lambda)$, being a rational function of λ, is determined by its values at large dominant λ. □

Because of this, we can write $A_{w,V} \stackrel{\text{def}}{=} A_{w,V,\delta}$, where δ is a reduced decomposition of w.

Definition 9.14 $A_{w,V}(\lambda)$ *are called the dynamical Weyl group operators.*

Proposition 9.15 *Let* $w_1, w_2 \in \mathbf{W}$, *with* $l(w_1 w_2) = l(w_1) + l(w_2)$. *Then,*

$$A_{w_1 w_2, V}(\lambda) = A_{w_1, V}(w_2 \cdot \lambda) A_{w_2, V}(\lambda).$$

Proof This follows directly from the definition of $A_{w,V}$. □

Definition 9.16 *The braid group* $\tilde{\mathbf{W}}$ *attached to* $\mathbf{W}$ *is generated by* s_i *and by the same relations as* $\mathbf{W}$*, with the exception of the relations* $s_i^2 = 1$*. Alternatively,* $\tilde{\mathbf{W}}$ *is the fundamental group of the open set in* $\mathfrak{h}$ *defined by the inequalities* $\langle \alpha, \mathsf{x} \rangle \neq 0, \alpha \in \mathsf{R}_+$.

Corollary 9.17 *There is a* $\mathbf{C}$*-linear action of* $\tilde{\mathbf{W}}$ *on the space of meromorphic functions of* λ *with values in* V*, which on generators is given by*

$$(s_i \circ f)(\lambda) \stackrel{\text{def}}{=} A_{s_i,V}(s_i \cdot \lambda) f(s_i \cdot \lambda).$$

This action is called the *shifted dynamical action*.

Remark 9.18 Note that $\mathbf{W}$ can be regarded as a subset (not a subgroup) of $\tilde{\mathbf{W}}$, by sending any $w \in \mathbf{W}$ to (any) reduced decomposition, regarded as an element of $\tilde{\mathbf{W}}$. For $w \in \mathbf{W} \subset \tilde{\mathbf{W}}$ the shifted dynamical action can be written as

$$(w \circ f)(\lambda) = A_{w,V}(w^{-1} \cdot \lambda) f(w^{-1} \cdot \lambda).$$

We can also define a modified action.

Definition 9.19 *The (unshifted) dynamical action of* $\tilde{\mathbf{W}}$ *on the space of meromorphic functions of* λ *with values in* V *is defined by*

$$(w \star f)(\lambda) = \mathcal{A}_{w,V}(w^{-1} \cdot \lambda) f(w^{-1} \cdot \lambda) \quad \textit{for all } w \in \mathbf{W} \subset \tilde{\mathbf{W}},$$

where $\mathcal{A}_{w,V} \stackrel{\text{def}}{=} A_{w,V}\left(-\lambda - \rho + \frac{1}{2}h\right)$.

Proposition 9.20 *On functions with values in* $V[0]$*, the dynamical action of* $\tilde{\mathbf{W}}$ *factors through* $\mathbf{W}$.

Proof See Etingof and Varchenko (2002). □

Lemma 9.21 *Let* U, V *be representations of* $\mathfrak{U}_\mathsf{q}(\mathfrak{g})$ *and let* $w \in \mathbf{W}$*. Then*

$$A_{w,U\otimes V}(\lambda) J_{UV}(\lambda) = J_{UV}(w \cdot \lambda) A^{(2)}_{w,V}(\lambda) A^{(1)}_{w,U}(\lambda - h^2).$$

Proof Similar to the proof of Lemma 9.5. □

Corollary 9.22 *Let U, V be representations of $\mathfrak{U}_{\mathsf{q}}(\mathfrak{g})$ and let $w \in \mathbf{W}$. Then,*

$$R_{UV}(w \cdot \lambda) = A^{(2)}_{w,V}(\lambda) A^{(1)}_{w,U}(\lambda - h^2) \mathcal{R}_{VU}(\lambda) \left(A^{(2)}_{w,V}(\lambda - h^1) \right)^{-1} A^{(1)}_{w,U}(\lambda)^{-1}.$$

Theorem 9.23 *The Macdonald–Ruijsenaars operators ($\mathbf{D}_W$) are invariant with respect to the dynamical action of $\mathbf{W}$; i.e.,*

$$[\mathbf{D}_W, w\star] = 0 \quad \textit{for all } w \in \mathbf{W}.$$

Proof Let $f : \mathfrak{h}^* \to V[0]$ be a meromorphic function. We have

$$\begin{aligned} &\left((w\star)^{-1} \mathbf{D}_W (w\star) f\right)(\lambda) \\ &= \mathcal{A}_{w,V}(\lambda)^{-1} \sum_{\nu \in \mathfrak{h}^*} (\mathsf{tr}|_{W[w\nu]} \mathbf{R}_{WV}(w \cdot \lambda)) \mathcal{A}_{w,V}(\lambda + \nu) f(\lambda + \nu). \end{aligned} \tag{9.4}$$

Now, corollary 9.22 implies that

$$\mathbf{R}_{WV}(w \cdot \lambda) = \mathcal{A}^{(2)}_{w,V}(\lambda) \mathcal{A}^{(1)}_{w,W}(\lambda + h^2) \mathbf{R}_{VW}(\lambda) \left(\mathcal{A}^{(2)}_{w,V}(\lambda + h^1) \right)^{-1} \mathcal{A}^{(1)}_{w,W}(\lambda)^{-1}. \tag{9.5}$$

Substituting (9.5) into the right-hand side of (9.4) and using the facts that f takes values in $V[0]$ and that trace is conjugation invariant, we see that the $\mathcal{A}$ cancel out, and we obtain

$$\left((w\star)^{-1} \mathbf{D}_W (w\star) f\right)(\lambda) = (\mathbf{D}_W f)(\lambda).$$

Thus $(w\star)^{-1} \mathbf{D}_W (w\star) = \mathbf{D}_W$, and the result follows. □

Theorem 9.24 *The function $F_V(\lambda, \mu)$ is invariant under the dynamical action of $\mathbf{W}$. In other words, for all $w \in \mathbf{W}$, we have*

$$F_V(\lambda, \mu) = F_V^w(\lambda, \mu),$$

where

$$F_V^w(\lambda, \mu) = \left(\mathcal{A}_{w,V}(w^{-1} \cdot \lambda) \otimes \mathcal{A}_{w,V^*}(w^{-1} \cdot \mu) \right) F_V(w^{-1} \cdot \lambda, w^{-1} \cdot \mu).$$

Proof By Theorem 9.23, we see that $F_V^w(\lambda, \mu)$ is a solution of the Macdonald–Ruijsenaars equations and the dual Macdonald–Ruijsenaars equations. Using arguments similar to those in the proof of the symmetry identity, we conclude that $F_V^w(\lambda, \mu) = c_w F_V(\lambda, \mu)$ for some constant c_w. The constant can be shown to be 1 by taking limits as $\mathsf{q}^{\langle \mu, \alpha_i \rangle} \to 0, \mathsf{q}^{\langle \lambda, \alpha_i \rangle} \to 0$. □

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